of the proposed By-Laws, which were adopted subject to the approval of the national organization.

The first regular meeting of the section was set for April 8, 1933, and an invitation for that time from Beloit College, proffered by Professor H. H. Conwell, was accepted.

It was decided that the temporary officers should function until the meeting in April, and that the temporary chairman should appoint a nominating committee to propose a list of officers which would be sent out by mail to the members shortly before the spring meeting.

Professor Conwell and Mr. Henry Ericson were elected to the program committee.

A vote of thanks was offered to Professor Parkinson and the Extension Division for their work and hospitality.

H. P. PETTIT, *Temporary Secretary*

THE EXPANSION OF DETERMINANTS OF COMPOSITE ORDER

By JOHN WILLIAMSON, Johns Hopkins University

It is often convenient, in discussing determinants of order mn, to consider the n^2 square arrays of order m contained in the determinant[1] as matrices of order m. In what follows we use this method to reduce[2] a determinant of order mn to a determinant of order $m(n-1)$. We first consider the special case in which $n=2$ and then pass on to the more general case. The final theorem proved is simply an extension of Horner's theorem[3] in which single elements of a determinant are replaced by matrices of order m.

Let Δ be the determinant of the square array

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

where A, B, C, D are all square matrices of order m whose elements are complex numbers, so that Δ is a determinant of order $2m$. We may denote this by

$$\Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix}.$$

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[1] H. W. Turnbull, *Determinants, Matrices and Invariants*, pp. 39 sq.

[2] A method of reducing a determinant of order mn to one of order $m(n-1)$, without the use of matrices, is given in Muir and Metzler, Theory of Determinants, p. 150.

[3] Muir, *History of the theory of determinants*, vol. 3, pp. 15–16.

With this notation

$$\Delta = (-1)^m \begin{vmatrix} B & A \\ D & C \end{vmatrix} = (-1)^m \begin{vmatrix} C & D \\ A & B \end{vmatrix} = \begin{vmatrix} D & C \\ B & A \end{vmatrix}, \tag{1}$$

for

$$\begin{vmatrix} B & A \\ D & C \end{vmatrix}$$

is obtained from Δ by interchanging the first set of m columns of Δ with the second set of m columns, while

$$\begin{vmatrix} C & D \\ A & B \end{vmatrix}$$

is obtained from Δ by a similar process applied to the rows of Δ. Further if E denotes the unit matrix of order m,

$$\begin{vmatrix} E & B \\ C & D \end{vmatrix} = |D - CB|, \tag{2}$$

a result which expresses a determinant of order $2m$ as a determinant of order m. To prove formula (2) we note that

$$\begin{vmatrix} E & B \\ C & D \end{vmatrix} = \begin{vmatrix} E & B \\ C - CE & D - CB \end{vmatrix},$$

since on the right the $(m+i)$-th row $(i = 1, 2, \cdots, m)$ is obtained by subtracting a linear combination of the first m rows from the corresponding row on the left. But

$$\begin{vmatrix} E & B \\ C - CE & D - CB \end{vmatrix} = \begin{vmatrix} E & B \\ 0 & D - CB \end{vmatrix}$$
$$= |E||D - CB| = |D - CB|,$$

where 0 denotes the zero matrix.

We are now in a position to prove the theorem:

THEOREM I. *If A is non-singular, then*

$$\Delta = |A||D - CA^{-1}B| \tag{3}$$

where A^{-1} is the matrix reciprocal to A.

By the multiplication theorem on matrices we have

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E & A^{-1}B \\ C & D \end{pmatrix}$$

and by taking determinants of both sides of this matrix equation, we obtain

$$|A^{-1}|\Delta = \begin{vmatrix} E & A^{-1}B \\ C & D \end{vmatrix},$$

$$= |D - CA^{-1}B| \text{ by (2)},$$

and this final result is equivalent to (3). Formula (3) may also be written in either of the forms

$$\Delta = |AD - ACA^{-1}B| \tag{4}$$

or

$$\Delta = |DA - CA^{-1}BA| . \tag{5}$$

From (4) it follows that, if A and C are commutative,

$$\Delta = |AD - CB| \tag{6}$$

and from (5) that

$$\Delta = |DA - CB| , \tag{7}$$

if A and B are commutative. If B is non-singular

$$\Delta = (-1)^m \begin{vmatrix} B & A \\ D & C \end{vmatrix} \text{ by (1)},$$

$$= (-1)^m |B||C - DB^{-1}A| \text{ by (3)},$$

$$= |B||DB^{-1}A - C|. \tag{8}$$

Similarly, if C is non-singular,

$$\Delta = (-1)^m |C||B - AC^{-1}D| = |C||AC^{-1}D - B| , \tag{9}$$

and, if D is non-singular,

$$\Delta = |D| \, |A - BD^{-1}C| . \tag{10}$$

If B and D are commutative, from (8)

$$\Delta = |DA - BC| \tag{11}$$

and from (9), if C and D are commutative,

$$\Delta = |AD - BC|. \tag{12}$$

Formula (3) is not in its most convenient form for the calculation of numerical determinants, as the adjugate matrix is usually more convenient for calculation than the reciprocal matrix. If a denotes the determinant of A, so that $|A| = a$, then the matrix $A^{-1} = a^{-1}A^*$, where A^* is the adjugate matrix of A. Accordingly (3) becomes

$$\Delta = a|D - CA^*Ba^{-1}| = a^{1-m}|Da - CA^*B| . \tag{13}$$

Let Δ now be the determinant of the square array A_{rs} $(r, s, = 1, 2, \cdots, n)$, where A_{rs} is a square matrix of order m, so that Δ is a determinant of order mn. We may denote the matrix of the array by (A_{rs}) and the determinant Δ by $|A_{rs}|$. If A_{ij} is non-singular, that is if the determinant of the matrix A_{ij} which we may denote by a is different from zero, the inverse matrix A_{ij}^{-1} exists. If (K_{rs}) is a matrix of the same type as (A_{rs}), where $K_{rs}=0$, $r\neq s$; $K_{rr}=E$, $r\neq i$; $K_{ii}=A_{ij}^{-1}$, by multiplication

$$(K_{rs})(A_{rs}) = (B_{rs}),$$

where $B_{rs}=A_{rs}$, $r\neq i$, $B_{ik}=A_{ij}^{-1}A_{ik}$. Moreover $|K_{rs}|=a^{-1}$ and therefore $\Delta=a|B_{rs}|$. But by reasoning similar to that used in the proof of formula (2),

$$|B_{rs}| = |C_{rs}|,$$

where $C_{rs}=A_{rs}-A_{rj}A_{ij}^{-1}A_{is}$, $r\neq i$,

$$C_{is} = A_{ij}^{-1}A_{is}.$$

But $C_{rj}=0$, $r\neq i$, and $C_{ij}=E$. Therefore by a formula similar to (1), we have the following theorem:

THEOREM 2. *If the m-rowed square matrix A_{ij} is non-singular and if Δ is the determinant of the nm-rowed square matrix (A_{rs}),*

$$\Delta = (-1)^{m(i+j)}a\,|C_{rs}| \qquad \begin{matrix} r = 1, 2, \cdots, i-1, & i+1, \cdots, n \\ s = 1, 2, \cdots, j-1, & j+1, \cdots, n, \end{matrix} \tag{14}$$

where

$$C_{rs} = A_{rs} - A_{rj}A_{ij}^{-1}A_{is}$$

and a is the determinant of the matrix A_{ij}.

Theorem 2 expresses a determinant of order mn as a determinant of order $m(n-1)$. It is a direct extension of Horner's theorem on the expansion of a determinant. In fact if $m=1$, that is, if all the matrices A_{rs} are matrices of one row and column, theorem 2 is exactly Horner's theorem.

Formula (14) may also be expressed in terms of the matrix adjugate to A_{ij} instead of in terms of the reciprocal of A_{ij}. If this is done formula (14) becomes

$$\Delta = (-1)^{m(i+j)}a^{1-m(n-1)}\,|D_{rs}|,$$

where $D_{rs}=aA_{rs}-A_{rj}A^{*}_{ij}A_{is}$ $r\neq i$, $s\neq j$, and A^{*}_{ij} is the matrix adjugate to A_{ij}.

So far we have been dealing with determinants, whose elements are complex numbers, but it is interesting to note the connection between this paper and one by A. R. Richardson, in which determinants whose elements are numbers of a division algebra are discussed.[1] This is best illustrated by a simple example. If A, B, C, D are elements of a division algebra the left hand determinant of

[1] A. R. Richardson, *Simultaneous Linear Equations Over a Division Algebra*, Proceedings of the London Mathematical Society, Series 2, Volume 28, pp. 395–430, (1928).

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

is defined as

$$DA - DBD^{-1}C \tag{15}$$

where D is not zero. If in this definition we regard A, B, C, D as matrices of order m and impose the restriction that D be non-singular, (15) may be regarded as the matrix-determinant of

$$\begin{matrix} A & B \\ C & D \end{matrix}.$$

Equation (10) is thus equivalent to the statement that the determinant of the matrix-determinant of the square array

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

is equal to the ordinary determinant of this array. A similar connection exists between theorem 2 and Richardson's definition of determinants of order n. It is obvious that, since the algebra of matrices is not a division algebra, the matrix-determinant defined by (15) is in itself of no practical importance.

A SIXTEENTH CENTURY CHINESE APPROXIMATION FOR π

By J. M. BARBOUR, Ithaca College

The standard work on Chinese mathematics is Yoshio Mikami's "The Development of Mathematics in China and Japan."[1] In it two chapters are devoted to circle-measurements. Recently Professor David Eugene Smith[2] has raised doubts concerning the authenticity of some of the supposedly ancient Chinese texts. He would ask us perhaps to accept with reservations the history of the earlier Chinese attempts to determine π, as he has excellently summarized them on page 309 of Volume 2 of his *History of Mathematics:*

"The value 3 was used probably as early as the 12th century B.C. and is given in the *Chôu-peï* and the *Nine Sections*. Ch' ang Höng [Hêng] (c. 125) used $10^{1/2}$, and Wang Fan (c. 265) used 142/45, which is equivalent to 3.155 . . . ." "Among other early Chinese values of no high degree of accuracy are those of Men (c. 575), who gave 3.14, and Wu (c. 450), whose value was 3.1432+." "Tsu Ch' ung-chih (c. 470) was able, by starting with a circle of diameter 10 feet, to obtain 3.1415927 and 3.1415926 for the limits of π, and from these, by

[1] Published as Vol. 30 of *Abhandlungen zur Geschichte der Mathematischen Wissenschaften*, 1913.

[2] *Unsettled Questions Concerning the Mathematics of China*, The Scientific Monthly, vol. 33, 1931, pp. 244–250.

interpolation, he obtained the 'accurate and inaccurate' values 355/113 and 22/7."

Mikami[3] adds that this "accurate value" was not known in Europe "until the Dutch mathematician Adriaan Anthoniszoon, father of Adriaan Metius, obtained it first in 1585." But Cajori[4] says that "M. Curtze's researches would seem to show that it [355/113] was known to Valentin Otto as early as 1573."

The later history of the computation of π in China was one of forgetting and rediscovering these values, and in turn supplanting them by others of no greater worth. Tsu's "inaccurate value," 22/7, reappears several times, as in Ch' êng Ta-wei's work, "Suan fa t' ung tsung" (Foundations of Arithmetic), 1593.[5] But his contemporary, Hsing Yün-lu, "at the close of the 16th century," gave the quite erroneous value, 3.126, and Ch'ên Chin-mo, "at the middle of the 17th century," used the almost as erroneous value, 3.15025.[6]

The date (1593) of Ch'êng's work is of interest in connection with the present study, as it is almost exactly contemporary with Prince Chu Tsai-yü's treatise on music, "Lü lü ching i," 1595[7] or 1596.[8] Tsai-yü is honored by musicians for having been one of the first persons—if not the first—to give the "exact lengths and bores of 12 pipes in correctly equal temperament."[9] The problem of finding these lengths involves the extraction of the 12th root of 1/2, for the ratio for each semitone is $1/2^{1/12}$. Tsai-yü gives the approximate lengths to nine places, as 10.00000000, 9.43874312, 8.90898718, etc.[10]

Unfortunately Tsai-yü does not explain how his no doubt very laborious calculations were made. Mersenne,[11] in solving the same problem to six places only, actually writes 2048 followed by 60 zeros as the number of which the 12th root is to be taken, to give the first fret of the viol; i.e., a number that is twice the second one of Tsai-yü's lengths, if we ignore the location of the decimal point. So we can picture Tsai-yü beginning with 108 zeros.

Not content with stating the lengths of the pipes, Prince Tsai-yü has furnished us also with their outer and inner diameters.[12] These are given to three places only. The inner diameter for the lowest pipe is .500, for its octave .353, and for its double octave, .250. Obviously the diameters vary as the square

[3] Op. cit., p. 50.

[4] Florian Cajori: *A History of Mathematics*, 2nd edition, 1919, p. 73.

[5] Moritz Cantor: *Geschichte der Mathematik*, vol. 1, p. 645.

[6] Mikami, op. cit., p. 138.

[7] Maurice Courant: *Chine et Corée*, in vol. 1 of Lavignac's "Encyclopédie de la Musique et Dictionnaire du Conservatoire," 1913, p. 210.

[8] Père Amiot: *De la musique des Chinois*, vol. 6 of "Mémoires concernant l'histoire, les sciences, les arts, les moeurs, les usages, & des Chinois," 1780, p. 33, note (2).

[9] Alexander J. Ellis: Appendix to Helmholtz's *Sensations of Tone*, 3rd ed., 1895, p. 784. Ellis incorrectly gives the date of Tsai-yü's work as 1573.

[10] Amiot, op. cit., Seconde Partie, Figure 18, Planche XXI. Amiot has made an error in transcribing the Chinese numbers, having inserted a superfluous zero in each one, between the fifth and sixth digits from the left as given above.

[11] Marin Mersenne: *Harmonie Universelle*, 1636/7, p. 38.

[12] Amiot, op. cit., p. 105–109.

roots of the lengths. V. Charles Mahillon[13] constructed a series of tubes in strict observance of these directions and found that this proportion of diameter to length gave the necessary correction for the pipes to be exactly in tune. He adds that in this respect "la théorie chinoise est en avance sur la nôtre, qui ne donne à ce sujet aucun renseignement."

For the sake of completeness, Tsai-yü has included tables of the areas and of the volumes of the twelve tubes of the middle octave—the tubes whose lengths were mentioned above.[14] The stated value (.353) for the diameter of the first tube is too inaccurate to use, for it is evident that these tables—the first to nine places and the second to fifteen—were constructed as carefully as was the table of lengths. If we replace the stated value by $2^{1/2}/4$, we have $\pi/32 = .0982092751$ (the area of the cross-section of the pipe). Thus $\pi = 3.1426968032$. Or we may use the volume of this pipe, .982092751647982, which is the area multiplied by 10.00000, the length of the pipe to six places. From this second figure we compute that $\pi = 3.1426968052735424$.

At first glance these values would seem to be a rude attempt at expressing $\pi = 22/7 = 3.142857 \cdots$. But it is easy to show that they represent something quite different. The area of the seventh pipe in this octave is, as given, .0694444444. This number undoubtedly represents 5/72. The length of this pipe will be $2^{1/4}/4$ (Tsai-yü has .297), and its area $\pi 2^{1/2}/64$. Hence $\pi 2^{1/2}/64 = 5/72$, and $\pi = 20(2^{1/2})/9$. This agrees with the value computed at the end of the previous paragraph until the last two places, which should be 45, an error due to the fact that the next two places of the area, after those given, will be 67.

This value for π is one that could have been obtained directly by comparing the lengths of the chord and the arc of $\pi/2$. As a practical value, it differs too slightly from the simpler expression, 22/7, to have been generally used in place of the latter. Its selection was probably due to number symbolism—in the sense that certain numbers have magical properties.

As an illustration of the same sort of thing, let us consider the value $\pi = 10^{1/2}$, previously mentioned. Smith[15] says, in writing of "Other Greek Approximations of π": "Since one of the common approximations for a square root in the Middle Ages was $(a^2+r)^{1/2} = a + r/(2a+1)$, and since this gives $10^{1/2} = 3\ 1/7$, it is natural to expect that $10^{1/2}$, which is $3.1623 \cdots$, would often have been given as the value of π, and this was in fact the case."

Cantor,[16] in referring to the same approximation, is somewhat dubious: "Das heisst doch: man ersetzte 3 1/7 durch $10^{1/2}$, einen rational Werth durch einen irrationalen, und das kommt in der ganzen Geschichte der Mathematik nirgends vor."

But Mikami[17] must have the final word, so far as China is concerned: "If we

[13] Quoted by Courant, op. cit., p. 86.
[14] Amiot, op. cit., Seconde Partie, Figure 19 and Figure 20, Planche XXII.
[15] *History of Mathematics*, vol. 2, p. 307.
[16] Op. cit., p. 607.
[17] Op. cit., p. 137.

don't err in our judgment, this value $\pi = 3.16$ or $10^{1/2}$ was adopted, perhaps because for so important a quantity as π it had seemed preferrable [sic] to look upon it as underlying a simple law of formation. Borrowed from abroad or not, [Cantor had traced it to India and Arabia] it was equally the love of simplicity of the Chinese mind that had worked to apply such a value as this in practical purposes."

In spite of Cantor's protest, one must agree that this is a "simple" value in the form in which it had been stated by Chang—"the square on the circular circumference and the perimeter of the circumscribed square are as 5 to 8."[18] But the quantity $10^{1/2}$, although less simple than 5/8, has a philosophical meaning as a function of 10—the foundation of the common number-system.

The very ancient approximation, $\pi = 3$, must have originated at a time when its degree of accuracy was entirely compatible with the methods of measurement used in that epoch. But its retention after these methods had improved was due not so much to inertia as to the veneration of the number 3. This Pythagorean cult may have been introduced by the army of Alexander, together with other Grecian ideas.[19]

In a quite different sphere this cult survives today in the "official" Chinese musical scale, which was reaffirmed in 1712[20] after centuries of use. Practically the scales in use today show great divergences from one another and from the norm as well.[21] But in theory the Chinese scale coincides with the Pythagorean scale of ancient Greece, using the ratio of 3:2 for the fifth.

Thus there have been in China these two co-existent number-systems—the 10-system and the 3-system, although the latter had early developed into a 9-system. In the ancient description of the dimensions of the fundamental musical pipe[22] one sees a beautiful example of this conflict between systems. The length of this pipe was 100 grains of millet (*chou*), joined at their smaller diameter, or 81 grains, if they were joined at their larger diameter. Its diameter was 3 grains, joined at the larger diameter. It contained exactly 1200 grains, or, since this pipe contained in itself, as it were, the twelve semitones of the octave in which it was the lowest tone, there were 100 grains for each semitone[23] —an interesting anticipation of Ellis' musical cents.[24]

The above illustrations prove nothing in connection with Tsai-yü's value for π. They have been introduced to show possible skeptics that it is not unreasonable to suppose that Tsai-yü was familiar with Ch'êng's approximation 22/7, but that he preferred the simplicity and the symbolism of the doctrine that, if the chord of the quarter-circle is 9, the arc thereof is 10.

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[18] Mikami, op. cit., p. 47.
[19] Édouard Chavannes: *Les Mémoires historiques de Se-ma Ts'ien.* Tom 3, Appendice II.
[20] Courant, op. cit., p. 92.
[21] Robert Lachmann: *Musik des Orients*, 1929, Chap. 1, Sect. 2. Instrumentalstimmungen.
[22] Amiot, op. cit., p. 85 ff.
[23] Amiot, op. cit., p. 92.
[24] Ellis, op. cit., Appendix XX, Section C.

Note by the Author. After the above article was in type, my attention was directed—through the courtesy of Dr. Yuen-ren Chao, director of the Chinese Educational Mission—to an article by Liu Fu in the special volume (commemorative of the 65th birthday anniversary of Ts'ai Yüan-p'ei) of the *Bulletin of the National Research Institute of History and Philology, The Academia Sinica*, Peiping, 1932, pp. 279–310.

Mr. Shao-wen Ling, a graduate student at Cornell, kindly assisted me in obtaining some idea of the contents of this article, the title of which may be translated "Chu Tsai-yü, Inventor of Equal Temperament." The first part of the article carefully explains the whole theory of acoustics and temperament, with copious references to such European authorities as Helmholtz, Ellis, et al. The history of temperament in Europe is taken from secondary sources, principally from Hugo Riemann.

Tsai-yü, his family, his life, and his work are then discussed exhaustively. He was born in 1536, being 60 years of age when his book on temperament appeared. He died after 1610, the year of publication of the last of his 18 volumes. Amiot and Courant are mentioned as having given a correct account of Tsai-yü's researches into equal temperament. Furthermore, a lengthy paragraph is devoted to Tsai-yü's value for π, expressed in decimal form (3.1426968) and compared with the similar value 22/7 and the closer approximation 355/113. There the parallelism with my article ends, for Tsai-yü's value is not given in its equivalent radical form $(20\ 2^{1/2})/9$.

BI-AFFINE GEOMETRY IN THE PLANE

By C. E. CLARK, Brown University

1. In the customary treatment of plane projective geometry[1] various other geometries are obtained by specializing certain elements. Specializing a "line-at-infinity," the sub-group of projective transformations for which this line is self-corresponding yields affine geometry, which contains Euclidean geometry as a special case. Affine and Euclidean geometries, however, lack a distinctive property of projective geometry, namely of being self-dual. In this paper we consider another specialization of affine geometry in which not only a "line-at-infinity" but also a point (the "origin") not incident to the given line is fixed.[2] This bi-affine geometry is self-dual. The transformations are those used in a homogeneous system such as a vector space, in which there is a unique zero vector that can not be transformed into any other finite vector.[3]

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[1] As for instance in Graustein, *Higher Geometry;* Veblen and Young, *Projective Geometry*; Forder, *Higher Course Geometry*; and Heffter and Koehler, *Lehrbuch der Analytischen Geometrie.*

[2] Although this is a type of transformation very frequently used in analytic applications, apparently the only discussion of the geometry of these transformations is that given by A. A. Bennett in the Annals of Mathematics, Vol. 27, 1925–26, p. 84.

[3] See Courant and Hilbert, *Methoden der Mathematischen Physik*, p. 1.

Consider as our plane the points with homogeneous coordinates (x_1, x_2, x_3),[4] where x_1, x_2, and x_3 are in a given field F[5]. As usual one defines a line, conic, etc., as the set of these points satisfying respectively a linear equation, a quadratic equation, etc., the coefficients of the equation in every case being in the given field F. The origin (0, 0, 1) and the points at infinity $(x_1, x_2, 0)$ are called *improper points*, the remaining points being *finite points*. By definition, a bi-affine point-point transformation is one that transforms finite points into finite points.

In section 2 we classify the bi-affine linear transformations, in section 3 reduce the second degree form to normal form, and in section 4 classify the conics, giving their normal forms in section 5.

2. A bi-affine linear transformation is of the form[6] $x_1 = aX_1 + bX_2$, $x_2 = cX_1 + dX_2$, $x_3 = X_3$, with a, b, c, and d in F, and $ad - bc \neq 0$. If $px_1 + qx_2 = 0$ is a self-corresponding[7] beam[8] for this transformation, $p(ax_1 + bx_2) + q(cx_1 + dx_2) \equiv \rho(px_1 + qx_2)$ for a fixed ρ in F, and $p(a - \rho) + qc = 0$, $pb + q(d - \rho) = 0$. It follows that the self-corresponding beams of the given transformation are in (1,1) correspondence with the values of ρ satisfying the quadratic equation

$$\begin{vmatrix} a - \rho & b \\ c & d - \rho \end{vmatrix} = 0$$

(unless $a = d$ and $b = c = 0$, in which case every beam is self-corresponding). Hence, in a field such as the complex field where every quadratic equation is reducible, every transformation has one or more self-corresponding beams. Furthermore, a transformation without self-corresponding beams in a given field F_1 may have self-corresponding beams in some extended field F_2 which contains F_1 as a sub-field. We now classify the bi-affine linear transformations, and give normal forms in which they may be expressed by proper choice of coordinate system.[9]

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[4] Although in most of our discussion non-homogeneous coordinates will suffice, we shall use homogeneous coordinates such as are customarily employed in analytic projective geometry.

[5] For a definition and the elementary properties of a number field see G. Scorza, *Corpi numerici e algebre*, Messina, 1921; or L. E. Dickson, *Algebras and their arithmetics*, Chicago, 1923.

If, in particular, F is the real field, our plane is the one customarily employed in elementary geometry; if F is the field of rational numbers, we consider only points whose coordinates are rational.

The question of whether or not certain numbers are squares is important in the following theory. This property of a number being a square depends upon the field; in the real field all positive numbers are squares and all negative numbers are not; in the rational field not all positive numbers are squares, e.g. 2 is not the square of a rational number; in the complex field every number is a square; in a Galois field $GF(p)$, half the non-zero numbers are squares and half are not.

[6] See footnote 1.

[7] Following the notation of Hudson's *Cremona Transformations* we say that a locus is *self-corresponding* if the transformation leaves the locus as a whole invariant, and that a locus is *invariant* if every point of the locus is invariant.

[8] A beam is a line that contains the origin.

[9] We may choose any coordinate system such that one vertex of the triangle of reference is the origin and the corresponding opposite side ($x_3 = 0$) is the line at infinity. This means that we may make a bi-affine transformation of coordinates.

(i) Every beam self-corresponding; called a *magnification* or *scalar transformation*; normal form $x_1 = a\ X_1,\ x_2 = a\ X_2,\ x_3 = X_3\ a \neq 0$. When $a = 1$ this is the identical transformation.

(ii) Exactly two self-corresponding beams; called a *stretch*; normal form $x_1 = a\ X_1,\ x_2 = b\ X_2,\ x_3 = X_3,\ a \neq b,\ ab \neq 0$. If $a = 1$ the stretch is a *simple stretch*, otherwise a *magnified stretch*.

(iii) Exactly one self-corresponding beam; called a *shear*; normal form $x_1 = a\ X_1,\ x_2 = b\ X_1 + a\ X_2,\ x_3 = X_3,\ a \neq 0$. If $a = 1$ the shear is a *simple shear*, otherwise a *magnified shear*.

(iv) No self-corresponding beam; called a *twist*; normal form $x_1 = X_2$, $x_2 = a\ X_1 + b\ X_2,\ x_3 = X_3,\ b^2 + 4a$ not a square.

3. By means of bi-affine linear transformations we now reduce to normal form the general second degree form $f(x_1, x_2, x_3) = \sum a_{ij} x_i x_j,\ i, j = 1, 2, 3,\ a_{ij}$ in F.

If $a_{11} = a_{12} = a_{22} = 0$, f is reducible either to $a_{33}\ x_3^2$, or to $x_3(x_2 + a_{33}\ x_3)$. Obviously a_{33} is an absolute bi-affine invariant.[1]

If at least one of $a_{11},\ a_{12},\ a_{22}$ is different from zero, then we can assume $a_{13} = 0$ and either $a_{11} = 0$ and $a_{23} \neq 0$ or $a_{11} = 1$. In the first case, when $a_{11} = 0$ and $a_{23} \neq 0$, if $\Delta = |a_{ij}| = 0$, f is reducible to either

$$\frac{2a_{22}}{a_{23}^2}\, x_2^2 + 2x_2x_3 + a_{33}x_3^2 \text{ or } x_1x_2 + x_2x_3,$$

while if $\Delta \neq 0$, f is reducible to

$$2x_1x_2 + 2x_2x_3 + a_{33}x_3^2,\ a_{33} \neq 0.$$

Finally, if $a_{13} = 0$ and $a_{11} = 1$, f is reducible to either

$$x_1^2 - \lambda x_2^2 + 2x_2x_3 + a_{33}x_3^2,\ \lambda = \frac{a_{12}^2 - a_{22}}{a_{23}^2},$$

$$\text{or } x_1^2 - \mu x_2^2 + a_{33}x_3^2,\ \mu = a_{12}^2 - a_{22}.$$

λ is an absolute bi-affine invariant of the class of forms considered in this paragraph (i.e. forms of the type $x_1^2 + a_{22}\ x_2^2 + a_{33}\ x_3^2 + 2a_{23}\ x_2x_3 + 2a_{12}\ x_1x_2$) and μ is a bi-affine invariant of weight two.

4. We now classify the non-singular conics.[2] We define a non-singular conic to be a hyperbola, parabola, or ellipse according as the locus contains two, one, or no points at infinity, and dually the non-singular conic is said to be hyper-

[1] For a complete system of bi-affine invariants of a set of forms see R. Weitzenböck, *Invariantentheorie*, Groningen, 1923, p. 232.

[2] For a Euclidean classification of conics see C. C. MacDuffee, *Euclidean invariants of second degree curves*, American Mathematical Monthly, vol. 33, 1926, p. 243.

bolic, parabolic, or elliptic according as two, one, or no tangents can be drawn to the curve from the origin. An origin-centered conic is one for which the origin is the pole of the line at infinity. It should be noted that according to this definition the same equation may give different types of conics in different fields. For instance, the equation of an ellipse in the real field will represent a hyperbola in the complex field, which latter field contains no ellipses.

Let $f(x_1, x_2, x_3)=0$ and $F(u_1, u_2, u_3)=0$ be the equations of the non-singular conic in homogeneous point coordinates and homogeneous line coordinates respectively. We recognize the following classes:

	$f(x_1, x_2, 0)$ having non-proportional factors	$f(x_1, x_2, 0)$ having proportional factors	$f(x_1, x_2, 0)$ irreducible
$F(u_1, u_2, 0)$ having non-proportional factors	hyperbolic hyperbola	hyperbolic parabola	hyperbolic ellipse
$F(u_1, u_2, 0)$ having proportional factors	parabolic hyperbola	parabolic parabola	parabolic ellipse
$F(u_1, u_2, 0)$ irreducible	elliptic hyperbola	elliptic parabola	elliptic ellipse

The class of hyperbolic hyperbolas falls into origin-centered and non-origin-centered loci; similarly for elliptic ellipses.

5. By means of the results of section 3, we can now give normal forms of the equations of the conics considered in section 4. Let λ, μ, and a_{33} have the values assigned them in section 3, and let $\Lambda=a_{33}(1+\lambda a_{33})$. Λ is an absolute invariant when λ is (see section 3). The normal forms are then:

hyperbolic hyperbola (non-origin-centered) (two distinct types) (i) $2x_1x_2+2x_2x_3+a_{33}x_3^2=0$, $a_{33}\neq 0$, and
(ii) $x_1^2-\lambda x_2^2+2x_2x_3+a_{33}x_3^2=0$, λ and Λ both squares $\neq 0$

parabolic hyperbola $\ldots\ldots x_1^2-\lambda x_2^2+2x_2x_3=0$, λ a square $\neq 0$

elliptic hyperbola $\ldots\ldots x_1^2-\lambda x_2^2+2x_2x_3+a_{33}x_3^2=0$, λ a square $\neq 0$, Λ not a square

hyperbolic ellipse $\ldots\ldots x_1^2-\lambda x_2^2+2x_2x_3+a_{33}x_3^2=0$, λ not a square, Λ a square $\neq 0$

parabolic ellipse $\ldots\ldots x_1^2-\lambda x_2^2+2x_2x_3=0$, λ not a square

elliptic ellipse (non-origin-centered) $\ldots x_1^2-\lambda x_2^2+2x_2x_3+a_{33}x_3^2=0$, λ not a square, Λ not a square

hyperbolic hyperbola (origin-centered) $\ldots x_1^2-x_2^2+a_{33}x_3^2=0$, $a_{33}\neq 0$

elliptic ellipse (origin-centered) $\ldots\ldots x_1^2-\mu x_2^2+a_{33}x_3^2=0$, μ not a square

hyperbolic parabola $\ldots\ldots x_1^2+2x_2x_3+a_{33}x_3^2=0$, a_{33} a square $\neq 0$

parabolic parabola $\ldots\ldots x_1^2+2x_2x_3=0$

elliptic parabola $\ldots\ldots x_1^2+2x_2x_3+a_{33}x_3^2=0$, a_{33} not a square.

A READING LIST IN THE ELEMENTARY THEORY OF EQUATIONS

By RAYMOND GARVER, University of California at Los Angeles

The following reading list is the result of a rather careful study of the five principal mathematical journals published in this country. It is hoped that it may serve two purposes. First, it may be used in several ways to supplement the usual work in a class in the theory of equations. The simpler articles (those of the list which are unstarred) may be put on an optional or required supplementary reading list, or made the basis of class reports, and in many cases several related articles can easily lead to a longer paper to be read before a club or society. It would probably be better, in most cases, not to assign articles for reading or study without a preliminary examination, since it is true that many of those which are not starred are actually not simple as to methods used. Several of the papers concerned with limits for roots of equations employ rather complicated proofs, for example, but are left unstarred because at least some of their results can be understood and applied by the average student.

The more difficult articles, which appear in the list with a star, are included principally for the instructors; but it is believed that they, too, may be of some value in class work, since the subjects are all related to the elementary theory of equations. Thus, the teacher who wishes to explain something of the nature of the various proofs of the fundamental theorem of algebra will find three papers on this topic, while a careful reading of two other starred articles will certainly give him a broader knowledge of symmetric functions.

The second main purpose of this bibliography is to acquaint the more enterprising students with the American mathematical periodicals, and to help them see how the first course in the theory of equations leads naturally, if not always immediately, into a number of advanced fields of study. Such a student who had found, let us say, the Gauss-Lucas polygon theorem of interest might be led to a study of Walsh's valuable work, and that of others, on Jensen's theorem, Pellet's theorem and more or less related topics. (References will be found at the end of Van Vleck's paper, which is No. 8 in the following list.) Another might be attracted to Curtiss's papers on extensions of Descartes' rule of signs, in volume 16 of the Transactions and volume 19 of the Annals; or to the recent work in elimination by Morley and Coble in volume 49 of the American Journal, and by Moore in volumes 30 and 31 of the Annals. Cramlet's article in volume 27 of the Annals, on determinants, might conceivably give a curious student the desire to know something about tensor analysis. And these are, of course, only examples. No one would claim that an undergraduate could read these more difficult papers at once or easily, but he might certainly get ideas which would help him outline his graduate work intelligently.

Lest possible misunderstanding arise, it should be stressed that the articles mentioned in the last paragraph are not given in this bibliography; they are merely samples of what a student who is familiar with the journals will find for

himself. The present list is restricted to papers whose subject matter is connected with a first course in the theory of equations. Only about half a dozen such papers have been omitted, and these seemed either misleading or of no possible value.

To save space, the different periodicals are represented by numerals, the American Journal of Mathematics by 1, the first and second series of the Annals of Mathematics by 2 and 3 respectively, the first and second series of the Bulletin of the American Mathematical Society by 4 and 5 respectively,[1] the Transactions of the American Mathematical Society by 6, and the American Mathematical Monthly by 7. Immediately after each title in the list appear, in order, the periodical number, the volume, the year of publication, and the paging. The brief comments which then follow, in certain cases, are designed merely to tell something of the nature of the article, when the title does not do this sufficiently, or to point out that part of the article may be difficult. They are not at all to indicate the more valuable papers. Finally, the first 19 volumes of the Monthly, through 1912, were not available to the writer of this paper. However, it is probably true that many libraries will not have these first volumes.

I. *Historical and General*[2]

1. Pierpont, Early history of Galois' theory of equations, 5, 4, 1898, 332–40. Largely a life of Galois.
2. White, Bézout's theory of resultants and its influence on geometry, 5, 15, 1909, 325–38.
3. Cajori, Horner's method of approximation anticipated by Ruffini, 5, 17, 1911, 409–14.
4. Miller, Definitions of the discriminant of a rational integral function of one variable, 7, 25, 1918, 287–90.
5. McClenon, A contribution of Leibniz to the history of complex numbers, 7, 30, 1923, 369–74. Remarks on Cardan's formulas for the cubic.
6. Miller, On the history of determinants, 7, 37, 1930, 216–19.
7. Funkhouser, A short account of the history of symmetric functions of the roots of equations, 7, 37, 1930, 357–65.

II. *Properties of Roots of Equations*

8. Van Vleck, On the location of roots of polynomials and entire functions, 5, 35, 1929, 643–83. Has a bibliography of 144 titles, of which 113 are on polynomials. Of these, 13 are listed in this paper, and 8 others appearing in American journals were examined but not included on account of their difficulty.
9.* Bôcher, Gauss's third proof of the fundamental theorem of algebra, 5, 1, 1895, 205–09.

[1] Series 1 (1892–4) was the Bulletin of the New York Mathematical Society.

[2] Under the main headings the grouping is partly chronological and partly by similar topics. If an article could appear under more than one heading a more or less arbitrary choice is made.

10.* Bôcher, Simplification of Gauss's third proof that every algebraic equation has a root, 1, 17, 1895, 266–68.

11.* Sheffer, A proof of the fundamental theorem of algebra, 5, 35, 1929, 227–30.

12. Coolidge, The continuity of the roots of an algebraic equation, 3, 9, 1908, 116–18.

13. Van Vleck, A sufficient condition for the maximum number of imaginary roots of an equation of the nth degree, 3, 4, 1903, 191–2. Condition is that a set of $n+1$ determinants are to be positive.

14. Kellogg, A necessary condition that all the roots of an algebraic equation be real, 3, 9, 1908, 97–98. Certain determinants must be negative.

15. Dunkel, Sufficient conditions for imaginary roots of algebraic equations, 3, 10, 1908, 46–54. A number of tests follow from a fundamental theorem

16. Dunkel, Generalized geometric means and algebraic equations, 3, 11, 1909, 21–32. An extension of paper 15 on conditions for imaginary roots. Also has a section on approximating roots.

17. Johnston, Real roots of a class of reciprocal equations, 7, 39, 1932, 415–18.

18. Carmichael and Mason, Note on the roots of algebraic equations, 5, 21, 1914, 14–22. Several theorems on limits of absolute values of roots. Some proofs rather complicated.

19. Birkhoff, An elementary double inequality for the roots of an algebraic equation having greatest absolute value, 5, 21, 1915, 494–95.

20. Carmichael, Elementary inequalities for the roots of an algebraic equation, 5, 24, 1918, 286–96. Good summary of results of papers 18 and 19 and of many papers in other periodicals. Also has new results, with some rather difficult proofs.

21. Williams, Note concerning the roots of an equation, 5, 28, 1922, 394–96. Modification of what is perhaps the main result of paper 18.

22. Walsh, An inequality for the roots of an algebraic equation, 3, 25, 1924, 285–86. A simple result, based, however, on a theorem in the Transactions.

23. James, On the upper limit to the real roots of an algebraic equation, 7, 34, 1927, 351–54.

24. Feld, An elementary upper bound to the roots of equations, 7, 37, 1930, 495–96.

25. Westerfield, New bounds for the roots of an algebraic equation, 7, 38, 1931, 30–35. Refers to papers 18, 20, 21 and others.

26. Roth, On algebraic equations having only real roots, 5, 38, 1932, 594–600.

27.* Mitchell, On the imaginary roots of a polynomial and the real roots of its derivative, 6, 19, 1918, 43–52. Gives limits for real and imaginary parts of complex roots.

28. Bôcher, Some propositions concerning the geometric representation of imaginaries, 2, 7, 1893, 70–72. Proof of the Gauss-Lucas convex polygon theorem, and its extension in the case of cubic equations.

29. Hayashi, Relation between the zeros of a rational integral function and its derivate, 3, 15, 1914, 112–13. Proof of Gauss-Lucas theorem.
30. Irwin, Relation between the roots of a rational integral function and its derivative, 3, 16, 1915, 138. Another proof of same theorem; see acknowledgment on page 146 of vol. 18.
31. Walsh, On the location of the roots of the derivative of a polynomial, 3, 22, 1921, 128–44. Walsh has other more advanced papers of the same general type in the Transactions and elsewhere.
32. Echols, Note on the roots of the derivative of a polynomial, 7, 27, 1920, 299–300. An algebraic proof of Jensen's theorem.
33. Bray, On the zeros of a polynomial and of its derivative, 1, 53, 1931, 864–72. A proof of a theorem formulated by Popovici.
34. McCulloch, Extension of Rolle's theorem, 2, 4, 1888, 5–8. The extended theorem is applied to the determination of the number of real roots of an equation.
35. Wright, Note on the practical application of Sturm's theorem, 5, 12, 1906, 346–7.
36. Fort, The Sturm and Fourier-Budan theorems and mixed differential-difference equations, 7, 33, 1926, 194–98.
37.* Murnaghan, A simple derivation of Waring's formulae, 7, 38, 1931, 219–22.
38. Gaines, A graphical method of deducing the criteria for the nature of the roots of cubic and quartic equations, 3, 1, 1900, 111–12.
39. Rees, Graphical discussion of the roots of a quartic equation, 7, 29, 1922, 51–55. Criteria for the nature of the roots.
40. Anning, A cubic equation of Newton's, 7, 33, 1926, 211–12. Discussion of a cubic arising in a geometrical problem.

III. *The Algebraic Solution of Equations*

41. Sawin, The rational functions of the cubic, 2, 9, 1895, 158–62. A solution of the cubic.
42. McClintock, A simplified solution of the cubic, 3, 2, 1901, 151–52. Employs a linear fractional transformation.
43. Oglesby, Note on the algebraic solution of the cubic, 7, 30, 1923, 321–23.
44. Frink, A method for solving the cubic, 7, 32, 1925, 134. Also gives a process for extracting cube roots of certain complex numbers.
45. Sawin, The algebraic solution of equations, 2, 6, 1892, 169–77. Covers both cubics and quartics.
46. Dixon, A new solution of biquadratic equations, 1, 1, 1878, 283–84.
47. Sawin, Solution of the quartic equation $x^4+Ax+B=0$, 2, 1, 1884, 14.
48. McClintock, On a solution of the biquadratic which combines the methods of Descartes and Euler, 5, 3, 1897, 389–90.
49. Cruchaga, Relating to a solution of the biquadratic equation, 7, 25, 1918, 29.

50. Oglesby, Solution of the general biquadratic, 7, 32, 1925, 250–51.
51. Smiley, A method of solving a biquadratic, 7, 35, 1928, 183–84.
52. Faà de Bruno, Résolution de la quintique dans le cas ou $I_{18}=0$, 1, 3, 1880, 162–63. Solution of a reciprocal quintic.
53. Haldeman, Resolution of a certain quintic equation and a geometrical construction of its roots, 7, 27, 1920, 257–58. Treatment of the De Moivre quintic.
54. Glenn, Relating to the quadratic factors of a polynomial, 7, 23, 1916, 313–15. Expecially quartics. Also extends synthetic division to the case where the divisor is quadratic.
55. Frumveller, Quadratic factors of polynomials, 7, 24, 1917, 208–12. An extension of paper 54. Considers quartic, quintic, sextic and general equations.
56. James, On the solution of algebraic equations with rational coefficients, 7, 31, 1924, 283–87. A discussion of quadratic factors.

IV. *The Geometrical Solution of Equations*

57. Mathews, Graphical constructions for imaginary intersections of line and conic, 7, 26, 1919, 447–51. Several graphical solutions of a general quadratic.
58. Gleason, A simple method for graphically obtaining the complex roots of a cubic equation, 3, 11, 1910, 95–96.
59. Irwin and Wright, Some properties of polynomial curves, 3, 19, 1918, 152–58. Gives graphical constructions for complex roots of quadratics, cubics and quartics. One of the constructions is given later by another author in the Monthly, vol. 25, 1918, 268–69.
60. Haldeman, Geometrical construction of the roots of a cubic, 7, 26, 1919, 390–92.
61. Ballantine, A graphic solution of the cubic equation, 7, 27, 1920, 203–04. A number of references are given.
62. Running, Graphical solutions of the quadratic, cubic, and biquadratic equations, 7, 28, 1921, 415–23. Gives a long list of references.
63. Graustein, A geometrical method for solving the biquadratic equation, 7, 35, 1928, 236–38.
64. Henderson, Observations on simultaneous quadratic equations, 7, 35, 1928, 337–46. Also gives Heilermann's geometrical solution of the quartic.
65. Henderson and Hobbs, The cubic and biquadratic equations, 7, 37, 1930, 515–21.
66. Bixby, Graphical solution of numerical equations, 7, 29, 1922, 344–46.
67. Dehn, Algebraic charts, 7, 39, 1932, 222–26.

V. *The Approximation of Roots*

68. Hamilton, The irreducible case of the cubic equation, 3, 1, 1899, 41–45. A numerical solution based on a table for $z^3-z+q=0$.

69. Gleason, On the complete logarithmic solution of the cubic equation, 3, 13, 1912, 120–22. Using trigonometric and hyperbolic functions. Notice errata at beginning of volume.
70. Capron, Relating to approximations to nearly equal roots of a cubic equation, 7, 25, 1918, 343–47.
71. James, An algebraically reducible solution of the cubic equation, 7, 32, 1925, 162–69. A method of approximation.
72. Pierce, An approximation to the least root of a cubic equation with an application to the determination of units in pure cubic fields, 5, 32, 1926, 263–69. First section only.
73. Risselman, On the solution of cubic equations, 7, 39, 1932, 229–30. A solution entirely in terms of hyperbolic functions.
74. Merriman, Final formulas for the algebraic solution of quartic equations, 4, 1, 1892, 202–05. Concerning the numerical solution of quartics with 2 real and 2 imaginary roots.
75. Merriman, The deduction of final formulas for the algebraic solution of the quartic equation, 1, 14, 1892, 237–45. Similar to paper 74.
76. Franklin, On Newton's method of approximation, 1, 4, 1881, 275–76. A slight modification of the usual presentation.
77. Kummell, On the method of continued identity, 2, 5, 1890, 85–98. An iterative method of approximation.
78. Ford, The solution of equations by the method of successive approximations, 7, 32, 1925, 272–87. A complete treatment. Newton's method is a special case.
79. Spenceley, On a method of approximating the real roots of a polynomial, 7, 32, 1925, 469–74. A method employing transformations of the given equation.
80. James, On the solution of higher degree algebraic equations, 5, 32, 1926, 162–65. Gives a recursion formula for the least positive root.
81. Camp, A method for accelerating the convergence in the process of iteration, 5, 33, 1927, 209–20. Will require rather serious study.
82. Deming, A method for approximating roots of algebraic equations in pairs, 7, 35, 1928, 364–67. A sextic equation is given as an example.
83. Pierce, An algorithm and its use in approximating roots of algebraic equations, 7, 36, 1929, 523–25.
84. Nygaard, Solution of equations by addition-subtraction logarithms, 7, 37, 1930, 486–91. Method applies to trinomial equations, algebraic or transcendental.
85. Kennedy, A new method of solving the equation $x^x = c$, 7, 38, 1931, 449–50. I have extended this method in 7, 39, 1932, 476–78.
86. Bouton, Discussion of a method for finding numerical square roots, 3, 10, 1909, 167–72. Newton's method applied to square roots and nth roots.
87. James, A rapid method of approximating arithmetic roots, 7, 31, 1924, 471–75.

88. Uspensky, Note on the computation of roots, 7, 34, 1927, 130–34. See also same volume page 366–68, and page 368–69.
89. Roman, Calculation of numerical roots, 7, 38, 1931, 320–22. Compare with various references under 88.
90. Moritz, Some physical solutions of the general equation of the nth degree, 3, 6, 1905, 64–78.[1] Gives five solutions, characterized as kinematic, visual, dynamic, hydrodynamic, electromagnetic.
91. Ponzer, An equation balance for class-room use, 7, 21, 1914, 283–85. Mechanical solution of cubics and quartics.
92. Rees, Relating to an equation balance, 7, 24, 1917, 136–37.
93. Candy, A mechanism for the solution of an equation of the nth degree, 7, 27, 1920, 195–99.

VI. *Determinants, Matrices, and Linear Equations*

94. Morley, Three notes on permutations, 4, 3, 1894, 142–48. On the method of teaching determinants.
95. Bennett, On the definition of determinants, 7, 31, 1924, 343–45.
96.* Murnaghan, The generalized Kronecker symbol and its application to the theory of determinants, 7, 32, 1925, 233–41. A new presentation of determinants.
97. Macloskie, A general method of evaluating determinants, 3, 6, 1904, 30. What is usually known as Chió's method of reduction. See also vol. 1, 1900, 74–76 of the Annals.
98. Heal, Expression of the coefficients of Sturm's functions as determinants, 2, 2, 1886, 85–87.
99. Van Vleck, On the determination of a series of Sturm's functions by the calculation of a single determinant, 3, 1, 1899, 1–13. See an article by Pell and Gordon, 3, 18, 1917, 188–93, for a slight modification of Van Vleck's work.
100. Coolidge, A simple algebraic paradox, 7, 21, 1914, 184–5. An exercise on Laplace's expansion. See page 327 of same volume for explanation in detail by Loria.
101. Pascal, On a certain class of determinants, 7, 22, 1915, 154–56. A generalization of paper 100, which is extended further by Metzler, 7, 25, 1918, 113 ff.
102. Moritz, On the cubes of determinants of second, third, and higher orders, 5, 18, 1912, 182–89.
103. Barnett, Real roots of equations with complex coefficients, 7, 31, 1924, 484–87. The method uses determinants.
104. Moulton, On the solutions of linear equations having small determinants, 7, 20, 1913, 242–49. Discussion of the accuracy of the solutions when the coefficients are furnished by observations.
105.* Burgess, Practical solution of linear equations, 7, 25, 1918, 441–44.

[1] Owing to incorrect paging, this is the second page 64 in vol. 6.

106. Ransom, Actual solution of simultaneous linear numerical equations, 7, 30, 1923, 316–18.
107. Deming, A systematic method for the solution of simultaneous linear equations, 7, 35, 1928, 360–63.
108. Kempner, Remarks on linear equations, 7, 36, 1929, 359–64.
109. Dines, On positive solutions of a system of linear equations, 3, 28, 1927, 386–92.
110. Ballantine, A graphical derivation of Cramer's rule, 7, 36, 1929, 439–41. A vector derivation, for systems of 2 or 3 equations.
111. Ballantine, Numerical solution of linear equations by vectors, 7, 38, 1931, 275–77.
112. Esty, Vectorial treatment of certain algebraic theorems, 7, 39, 1932, 338–47. Treats linear equations, and several other topics.
113. Ballantine, Note on the solution of a set of linear equations, 7, 31, 1924, 341. Furnishes a check if only one unknown is needed.

VII. *Miscellaneous*

114. Dines, Linear inequalities and some related properties of functions, 5, 36, 1930, 393–405. Largely expository. Refers to earlier papers on linear inequalities by Lovitt, 7, 23, 1916, 363–66; Carver, 3, 23, 1922, 212–20; Dines, 3, 20, 1919, 191–99 and 3, 28, 1926, 41–42. A later paper by Stokes, 6, 33, 1931, 782–805, is somewhat more complicated.
115. Schlauch, Mixed systems of linear equations and inequalities, 7, 39, 1932, 218–22.
116. Pierce, The practical evaluation of resultants, 7, 39, 1932, 161–62.
117.* Sayre, The solution of algebraic equations by partial differential equations, 3, 10, 1909, 116–22.
118. Dederick, Construction of an algebraic equation with an irrational root approximately equal to a given value, 7, 23, 1916, 69–71. See page 211 of the same volume for a problem on the same subject.
119.* Dresden, On symmetric forms in n variables, 3, 24, 1923, 227–36, and 3, 25, 1923, 71–84. A general treatment of symmetric functions.
120.* Bennett, Linear operations and generalized elementary symmetric functions, 7, 30, 1923, 180–85.
121. Field, On the theory of equations from the standpoint of vector analysis, 7, 32, 1925, 461–62.

I may also add that papers of my own in the Monthly, vols. 34–39, and in the Annals, vol. 29, may be of some interest. Of the 121 papers listed, 65 are from the Monthly, 29 from the Annals, 20 from the Bulletin, 6 from the American Journal and 1 from the Transactions. A number of solved problems in the Monthly are also worth consulting, among them vol. 22, pages 132 and 230, vol. 23, page 254, 24–131 and 232, 25–124 and 218, 26–291, 32–266, 33–229, 36–106, 37–317 and 554, 39–432.

CURVES DETERMINED BY A ONE-PARAMETER FAMILY OF TRIANGLES

J. H. WEAVER, Ohio State University

Some years ago the author published a paper[1] setting forth certain loci associated with a triangle which had its circumcircle and nine-point circle fixed. The results were obtained by the methods of elementary synthetic and analytic geometry. Later Prof. Murnaghan showed[2] that absolute coordinates could be used to good advantage in the proofs of these same properties. In the present paper absolute coordinates are used to prove some of the results in the paper cited above, and also to develop some new theorems relating to the same configuration.

In this discussion $A_i (i=1, 2, 3)$ will denote the vertices of the triangle, H the orthocenter and O the circumcenter. Let O be the origin. Then if the circumcircle is taken as the unit circle, the points A_i may be represented by $z_i = t_i$, where t_i is a turn, i.e., $|t_i| = 1$. Let $\bar{z}$ be the conjugate of z. Then without any difficulty it may be shown that the map equation of a straight line through the extremities of the vectors z_1 and z_2 may be written[3]

$$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) = z_2\bar{z}_1 - z_1\bar{z}_2. \tag{1}$$

Using this notation the following equations may be written. The side A_iA_j of the triangle

$$z + \bar{z}t_it_j = t_i + t_j. \tag{2}$$

In all the equations that follow, $i=1, 2, 3, j=1, 2, 3, k=1, 2, 3, i \neq j \neq k$. Altitude from the vertex A_k,

$$z - \bar{z}t_it_j = (t_k^2 - t_it_j)/t_k. \tag{3}$$

Tangent to the circumcircle at A_i

$$z + \bar{z}t_i^2 = 2t_i. \tag{4}$$

For a proper choice of the square root, the equation of the internal bisector of the angle A_i

$$z + \bar{z}t_i\sqrt{t_jt_k} = t_i + \sqrt{t_jt_k}. \tag{5}$$

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[1] *A System of Triangles Related to a Poristic System*, American Mathematical Monthly, vol. 31, 1924, p. 337.

[2] *Notes on Mr. Weaver's Paper*, American Mathematical Monthly, vol. 32, 1925, p. 37.

[3] In this connection see F. Morley, *Metric Geometry of the Plane n-line*, Transactions of the American Mathematical Society, vol. 1, 1900, p. 97: F. D. Murnaghan, loc. cit.: O. J. Ramler, *Loci of Some One-Parameter Systems of Lines*, American Mathematical Monthly, vol. 37, 1930, p. 130: H. A. DoBell, *On The Geometry of the Triangle*, American Mathematical Monthly, vol. 39, 1932, p. 71: A. R. Forsythe, *Complex Variables in Plane Geometry*, Quarterly Journal of Mathematics, vol. 42, 1911, p. 1.

For the same choice of square root as in (5) the equation of the external bisector of A_i

$$z - \bar{z}t_i\sqrt{t_jt_k} = t_i - \sqrt{t_jt_k}. \tag{6}$$

The polar of a point Z_1 with respect to the circumcircle

$$z\bar{z}_1 + \bar{z}z_1 = 2. \tag{7}$$

Side of the medial triangle parallel to A_iA_j[4]

$$z + \bar{z}t_it_j = (t_k^2 + \sigma_2)/(2t_k). \tag{8}$$

The line in which the external bisectors of the angles of the tangential triangle meet the opposite sides

$$z\sigma_2/\sigma_3 + \bar{z}\sigma_1 = 6. \tag{9}$$

Polar line of the orthocenter of the tangential triangle with respect to that triangle

$$z(\sigma_1^2 - 2\sigma_2) + \bar{z}(\sigma_2^2 - 2\sigma_1\sigma_3) = \sigma_1\sigma_2 - 3\sigma_3. \tag{10}$$

Points may be represented by vectors as follows:

Orthocenter

$$z = \sigma_1. \tag{11}$$

Foot of the altitude on A_iA_j

$$z = \tfrac{1}{2}(\sigma_1 - \sigma_3/t_k^2). \tag{12}$$

Mid-point of A_iA_j

$$z = \tfrac{1}{2}(t_i + t_j). \tag{13}$$

Intersection of the tangents to the circumcircle at the points A_i and A_j

$$z = 2t_it_j/(t_i + t_j). \tag{14}$$

Reflection of 0 in A_iA_j

$$z = t_i + t_j. \tag{15}$$

Symmedian point

$$z = \frac{2(\sigma_2^2 - 3\sigma_1\sigma_3)}{\sigma_1\sigma_2 - 9\sigma_3}. \tag{16}$$

Circumcenter of the tangential triangle

$$z = \frac{2\sigma_1\sigma_3}{\sigma_1\sigma_2 - \sigma_3}. \tag{17}$$

[4] The σ_i are the ordinary symmetric functions of the three t's, i.e., the t's are the roots of the equation $t^3 - \sigma_1t^2 + \sigma_2t - \sigma_3 = 0$.

Orthocenter of the tangential triangle

$$z = \frac{2(\sigma_2^2 - \sigma_1\sigma_3)}{\sigma_1\sigma_2 - \sigma_3}. \tag{18}$$

Nine-point center of the tangential triangle

$$z = \frac{\sigma_2^2}{\sigma_1\sigma_2 - \sigma_3}. \tag{19}$$

Incenter with proper choice of square roots

$$z = \sqrt{t_1t_2} + \sqrt{t_2t_3} + \sqrt{t_1t_3}. \tag{20}$$

Gergonne point of the tangential triangle

$$z = \frac{2(\sigma_2^2 - 2\sigma_1\sigma_3)}{\sigma_1\sigma_2 - 9\sigma_3}. \tag{21}$$

Nagel point of the tangential triangle

$$z = \frac{2(\sigma_2^2 + \sigma_1\sigma_3)}{\sigma_1\sigma_2 - \sigma_3}. \tag{22}$$

Certain Loci. Let us now assume that the orthocenter as well as the circumcircle remains fixed. We then have from (11) $\sigma_1 = c$ (c a constant) or $t_i + t_j = c - t_k$. If the line OH is chosen as the axis of reals then $c = \bar{c}$. The equation $\sigma_1 = c$ implies the conjugate equation $\sigma_2/\sigma_3 = c$. Hence we have the two equations

$$t_i + t_j = c - t_k \text{ and } t_it_j = \frac{t_k(c - t_k)}{ct_k - 1}. \tag{23}$$

Equations (23) show that t_i and t_j depend on t_k and are uniquely determined when t_k is chosen. It should also be noted that σ_3 is a variable turn since the product of any number of turns is another turn.

Equation (12) may now be written, by virtue of (23)

$$z = c/2 - \sigma_3/2t_k^2$$

which shows that the feet of the altitudes traverse a circle, namely the nine-point circle which is fixed in this system.

Equation (13) becomes

$$z = c/2 - t_k/2$$

which also represents the nine-point circle.

It may be shown in a similar fashion that the midpoint of the segment from the orthocenter to a vertex traverses the nine-point circle. These results are well known and are merely set down here to illustrate the method.

Equation (14) by virtue of (23) becomes

$$z = t_i/(ct_k - 1)$$

which is a circle unless $c = \pm 1$, when the locus is the tangent to the circumcircle at the orthocenter, which is on the circumcircle, together with the line at infinity. Hence in this system the tangential triangle is inscribed in a circle unless the orthocenter lies on the circumcircle, in which case the vertices of the tangential triangle lie on the tangent to the circumcircle at the orthocenter.

It should be noted that the case $c = \pm 1$ can occur only when the triangle $A_1A_2A_3$ is a right triangle, with $t_i = -t_j$ and $t_k = c = \pm 1$.

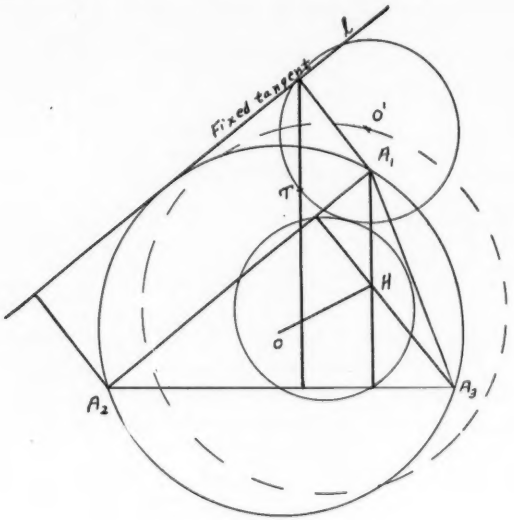

FIG. 1

The orthopole of a fixed tangent to the circumcircle is given by the equation[5]

$$z = \tfrac{1}{2}(\sigma_1 + 2t + \sigma_3/t^2) \quad \text{(where } t \text{ is a fixed vector).}$$

This is of the form $z = a + bt$ and is therefore a circle with the center at the point $(c+2t)/2$ and with radius $1/2$. Hence the locus of the orthopole of a fixed tangent to the circumcircle of the system is a circle equal to the nine-point circle. The center of this circle lies on a circle concentric with the nine-point circle and which has a radius equal to the radius of the circumcircle. We therefore conclude that the locus is also tangent to the nine-point circle (See Fig. 1, where l is the fixed tangent and the circle O' is the locus of the orthopole T of l).

Consider the point $P = t_0$. The equation of the line OP is

$$z - \bar{z}t_0^2 = 0.$$

[5] O. J. Ramler, loc. cit.

The equation of a tangent to the nine-point circle perpendicular to OP is

$$z + \bar{z}t_0^2 = (c + ct_0^2 - 2t_0)/2.$$

The intersection of these two lines is the point

$$z = (c + ct_0^2 - 2t_0)/4. \tag{24}$$

If t_0 is allowed to vary, this point traverses a limacon.[6] Let us now write

$$z = (c + ctt_0 - t - t_0)/4 \tag{25}$$

and let t be fixed. Equation (25) then represents a circle. For $t=t_0$ this circle has a point in common with the limacon (24). Moreover the limacon and circle are tangent at this point. For if we differentiate (24) and (25) with respect to t_0 we obtain from (24)

$$dz/dt_0 = (ct_0 - 1)/2$$

and from (25)

$$dz/dt_0 = (ct_0 - 1)/4.$$

These two derivatives represent vectors parallel to the two curves at the point $t=t_0$, and since these vectors are parallel the tangents must coincide. If in (25) t is considered as a parameter, we have a family of circles. The centers of the circles of this family will lie on the circle $z_1=(c-t)/4$ and the radii of the circles of the family will be equal to the absolute value of $(ct-1)/4$ which is the same as the absolute value of $(c-t)/4$. Hence all the circles of the family pass through the circumcenter of the family of triangles, and the limacon (24) is the envelope of the family of circles (25). The common point of the family of circles is the node of the limacon.

Equation (16) tells us that the locus of the symmedian point is a circle unless $c=\pm 3$ when the locus is at infinity. Equation (15) shows that the locus of the reflection of O in A_iA_j is a circle equal to the circumcircle and having its center at the orthocenter. Equation (17) states that the circumcenter of the tangential triangle is a fixed point which is at infinity if $c=\pm 1$. Equations (18), (19), (21), and (22) show that the respective loci are circles, while (20) leads us to a quartic. The proofs for these statements are simple and follow the same lines as those given above.[7]

Certain envelopes. Equation (2) by virtue of (23) becomes

$$z + \frac{\bar{z}t_k(c - t_k)}{ct_k - 1} = c - t_k. \tag{26}$$

Partial differentiation of (26) with respect to t_k gives for its envelope

[6] F. Morley, loc. cit.

[7] Equations 17–22 were developed by Miss Olive Givin, a student of the author.

$$\bar{z} = \frac{(ct_k - 1)^2}{ct_k^2 + c - 2t_k} \tag{27}$$

which is in general a conic, an ellipse if $c<1$, a hyperbola if $c>1$, and indeterminate if $c=1$. The foci are the circumcenter and the orthocenter.

Now write

$$\bar{z} = \frac{(ct_k - 1)(ct - 1)}{ctt_k + c - t_k - t} \quad \text{(where } t \text{ is fixed)}. \tag{28}$$

This is a straight line and considerations such as those applied to equations (24) and (25) show that (28) represents, for t as a parameter, the family of tangents to the conic (27). It may be shown in a similar fashion that the envelope of the medial triangle of the system is a conic.

Consider now a fixed point, $P=t$, on the circumcircle. The Simson Line of this point is

$$tz + \bar{z}\sigma_3 = \frac{t^3 + t^2\sigma_1 - t\sigma_2 - \sigma_3}{2t}.$$

The partial derivative of this with respect to t_k gives

$$z = (c - t)/2.$$

Hence the envelope of the Simson Lines of a fixed point on the circumcircle is a fixed point on the nine-point circle, and the radius of the nine-point circle to this point is parallel to the radius of the circumcircle to the given point.

From (13) we have that the midpoint of A_iA_j is

$$z = (c - t_k)/2.$$

The polar of this point with respect to the circumcircle is, because of (7),

$$\frac{z(ct_k - 1)}{t_k(c - t_k)} + \bar{z} = \frac{4}{c - t_k}.$$

The partial derivative of this equation with respect to t_k will determine the envelope of polars which is

$$z = \frac{4t_k^2}{ct_k^2 + c - 2t_k}.$$

This is the equation of an ellipse if $c<1$, a parabola if $c=1$, and a hyperbola if $c>1$.

The vertices of the conic, for $c\neq 1$, are $2/(c+1)$ and $2/(c-1)$. The foci are the circumcenter and the point $4c/(c^2-1)$. In the case of the parabola the vertex coincides with the orthocenter, which coincides with the vertex t_k, of the triangle, and the focus is the circumcenter.

Consider the external bisectors of the angles of the tangential triangle. These meet the opposite sides of this triangle in the points of the line

$$cz + c\bar{z} = 6$$

which is a fixed line. Hence the locus of the intersections of the external bisectors of the angles of the tangential triangle with the opposite sides is a fixed line.

Numerous other loci and envelopes may be determined by this method. For example, no mention is made here of the points and lines of the anti-medial triangle, and the Brocard configuration has been left untouched. However, it is hoped that enough examples have been given to show the beauty and power of the method.

QUESTIONS, DISCUSSIONS, AND NOTES

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A GEOMETRIC PARADOX—PROPOSED BY C. H. ROWE[1]

REPLIES BY J. L. COOLIDGE AND B. Z. LINFIELD

The Paradox

Consider the quadric surface that is represented by the general equation of the second degree in four homogeneous coordinates. It is well known that we must impose three independent conditions on the coefficients in order that the quadric should reduce to a pair of planes. However the following argument seems to show that two conditions are sufficient.

By imposing the condition that the discriminant of the equation should vanish we ensure that the quadric is degenerate, and that we can transform to a new system of homogeneous coordinates so that the transformed equation contains only three of the four variables. One further condition is sufficient to ensure that this transformed equation represents a pair of planes. The total number of conditions that we need is therefore two.

1. *A Reply by J. L. Coolidge, Harvard University.*

Professor Rowe's geometric paradox may be explained as follows. A quadric surface consists of two planes if the rank of the discriminant matrix is exactly two. That imposes on the coefficients three distinct equations of condition and certain inequalities besides. If the discriminant determinant is 0, the problem

[1] This paradox was proposed by Professor Rowe in the June-July issue of this MONTHLY, vol. 39 (1932), p. 352.

of finding the vertex of the supposed cone is determinate only if you rule out the case desired, by assuming the rank to be three and not two or else assume some extraneous condition. The total number of conditions is three as before.

2. *A Reply by B. Z. Linfield, University of Virginia.*

Professor Rowe's recent geometric paradox can be explained by the elementary fact that

The n^2 conditions for the square matrix A of order n to be zero, i.e. for $A_{ij}=0$ $(i,j=1,2,\cdots,n)$, are equivalent to the single condition that the quadratic function in x

$$x\cdot A\cdot x \equiv \sum_{i,j=1}^{n} A_{ij}x_ix_j$$

be identically zero for all values of x.[1]

For, let the homogeneous equation of the quadric be

$$x\cdot A\cdot x \equiv \sum_{i,j=1}^{4} A_{ij}x_ix_j = 0 \quad (A_{ij}=A_{ji}, \text{ i.e. } A=trnA).$$

If its discriminant be zero, i.e. det $A=0$, then (on expressing Professor Rowe's geometric argument in vector-matric equations) there exists a square matrix B of order 4, (det $B\neq 0$) such that under the linear transformation $x=y\cdot B$, we get

$$x\cdot A\cdot x = y\cdot BAB'\cdot y \quad (B'\equiv trnB),$$

and the symmetric matrix BAB' has its 4th row and column all zeros, i.e.

$$BAB' = \begin{Vmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & C_{22} & C_{23} & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}.$$

Therefore the quadratic function $y\cdot BAB'\cdot y$ will then not contain y_4, and it will be factorable (and therefore $x\cdot A\cdot x$ will be factorable) if only

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[1] Just as the n conditions that the vector $u=(u_1, u_2, \cdots, u_n)$ be zero, i.e. that $u_1=u_2=\cdots=u_n=0$, are equivalent to the single condition that the scalar product

$$u\cdot x \equiv u_1x_1 + u_2x_2 + \cdots + u_nx_n$$

be identically zero for all values of the vector x.

The notation used here is that if A is a rectangular matrix of n rows and m columns, then $A_1, A_2, A_3, \cdots$ designate its rows (vectors), and

$$x\cdot A \equiv A\cdot x = x_1A_1 + x_2A_2 + \cdots + x_nA_n,$$
$$x\cdot A_k \equiv A_k\cdot x = A_{k1}x_1 + A_{k2}x_2 + \cdots + A_{km}x_m,$$
$$x_iA_j \equiv A_jx_i = (A_{j1}x_i, A_{j2}x_i, \cdots, A_{jm}x_i),$$

and thus $x\cdot A$ is a vector linearly dependent on the vectors (rows) of the matrix A.

$$\det C \equiv \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = 0.$$

Hence the *two* conditions $\det A = 0$ and $\det C = 0$ are sufficient conditions that the quadric $x \cdot A \cdot x = 0$ degenerate into planes, while, as Professor Rowe points out, "it is well known that we must impose three independent conditions on the coefficients in order that the quadric should reduce to a pair of planes."

In fact, in order that $\det C = 0$ we must have rank $BAB' \leqq 2$. And, since the ranks of the matrices BAB' and A are equal, we must have rank $A \leqq 2$, i.e. $\operatorname{adj} A = 0$, and this alone involves three independent conditions (implying however $\det A = 0$).

The explanation of this difference in the number of conditions required that the quadric should degenerate into planes lies in the fact that $\det C$ is not a function of A alone. That, on the contrary,

$$\det C = z \cdot adj A \cdot z,$$

where the vector z is arbitrary. And that therefore the condition $\det C = 0$ implies the identity $z \cdot \operatorname{adj} A \cdot z = 0$ for all values of z which we have already seen means $\operatorname{adj} A = 0$.

For, to find the matrix B in the transformation $x = y \cdot B$, we have, since $\det A = 0$, that the four linear homogeneous equations in $x \cdot A = 0$ have a solution $b \cdot A = 0$ and $b \neq 0$. Now for *any* matrix D of the form

$$D \equiv \begin{Vmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \end{Vmatrix},$$

and

$$B \equiv \begin{Vmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ b_1 & b_2 & b_3 & b_4 \end{Vmatrix},$$

we have, no matter what be the values of the terms in D, that the product

$$BAB' = \begin{Vmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & C_{22} & C_{23} & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}, \quad DAD' \equiv C = \begin{Vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{Vmatrix},$$

and

$$\det C = |DAD'| \equiv |D| \cdot adjA \cdot |D|,$$

where $|D|$ is the vector orthogonal to each of the vectors (rows) D_1, D_2, D_3 of D, and of magnitude $|DD'|^{1/2}$. Hence, since the terms in the matrix D are arbitrary, (except that det $B \neq 0$), the value of the vector $|D|$ is perfectly arbitrary. And the only way det C can be zero for all values of $|D|$ is for adj $A = 0$.

The identity $|DAD'| \equiv |D| \cdot \text{adj } A \cdot |D|$ is independent of the order of the square matrix A, and can be used to explain analogous paradoxes for conics or hyperconicoids.

Thus, for the conic

$$x \cdot A \cdot x \equiv \sum_{ij=1}^{3} A_{ij} x_i x_j = 0$$

it is well known that its coefficients must satisfy three independent conditions in order that it shall represent one line, i.e. in order that $x \cdot A \cdot x$ should be a perfect square. However, if we impose the one condition that det $A \equiv |A| = 0$, we can find a transformation

$$x = y \cdot B, \text{ where } B_3 \cdot A = 0, \text{ and } |B| \neq 0,$$

such that

$$x \cdot A \cdot x \equiv y \cdot BAB' \cdot y,$$

and the quadratic function $y \cdot BAB' \cdot y$ will not contain y_3, i.e. the matrix

$$BAB' = \begin{Vmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & 0 \end{Vmatrix}.$$

Then the second condition,

$$|C| \equiv \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} = 0,$$

is sufficient to make $y \cdot BAB' \cdot y$ a perfect square, i.e. sufficient to insure the existence of a vector d such that $y \cdot BAB' \cdot y = (d \cdot y)^2$, and therefore $x \cdot A \cdot x = [(d \cdot B'^{-1}) \cdot x]^2$. Hence two conditions (det $A = 0$ and det $C = 0$) are sufficient to make $x \cdot A \cdot x$ a perfect square!

Obviously the explanation lies in the fact that the matrix D of the first two vectors (rows) of B is arbitrary (except that $|B| \neq 0$), that

$$C = DAD' \text{ and } |C| = |D| \cdot adjA \cdot |D|,$$

and therefore $|C| = 0$ requires adj $A = 0$. (This vector $|D|$ is sometimes called the "vector product" of D_1 by D_2 and designated by $D_1 \times D_2$, which is here the same as $B_1 \times B_2$ because $D_1 = B_1$ and $D_2 = B_2$.)

On the other hand, here is a paradox à-la-Coolidge about conics requiring a different explanation.

It is well known that *two* independent conditions must be imposed on a (real) central conic

$$F(x) \equiv A_{11}x_1^2 + 2A_{12}x_1x_2 + A_{22}x_2^2 + 2fx_1 + 2gx_2 + c = 0$$

in order that it should become a circle (namely $A_{12}=0$, $A_{11}=A_{22}$). Still, we can find a linear transformation $x=u+y\cdot B$ such that

$$F(x) \equiv t_1y_1^2 + t_2y_2^2 + F(u),$$

where t_1 and t_2 are the roots of the characteristic equation

$$|A - t\delta| \equiv \begin{vmatrix} A_{11}-t & A_{12} \\ A_{21} & A_{22}-t \end{vmatrix} = 0, \ \operatorname{trn} A = A,$$

of the matrix A. Then the *single* condition $t_1=t_2$ is sufficient to make this conic a circle!

Moreover, t_1 and t_2 here are clearly functions of the coefficients of $F(x)$ only; and therefore a single condition upon t_1 and t_2 is a single condition upon the coefficients of $F(x)$!

The explanation here lies in the fact that the condition that the roots of the characteristic equation

$$|A - t\delta| \equiv t^2 - (A_{11} + A_{22})t + |A| = 0$$

be equal is

$$(A_{11} + A_{22})^2 - 4|A| \equiv (A_{11} - A_{22})^2 + 4A_{12}^2 = 0 \quad (A_{21} = A_{12}),$$

which, because of the reality of the coefficients of $F(x)$, amounts to the two conditions $A_{11}-A_{22}=A_{12}=0$.

Similarly, it is well known that *five* independent conditions must be imposed on a (real) central quadric

$$F(x) \equiv x\cdot A\cdot x + 2b\cdot x + c = 0$$

in order that it should become a sphere, namely

$$A_{12} = A_{23} = A_{31} = 0 \text{ and } A_{11} = A_{22} = A_{33}.$$

Still, we can find a transformation (translation and rotation) $x=u+y\cdot B$ such that

$$F(x) \equiv t_1y_1^2 + t_2y_2^2 + t_3y_3^2 + F(u),$$

where t_1, t_2, t_3, are the roots of the characteristic equation

$$|A - t\delta| \equiv -(t^3 - t^2 \operatorname{div} A + t \operatorname{div}^2 A - |A|) = 0,$$

and

$$\operatorname{div} A = A_{11} + A_{22} + A_{33}, \quad \operatorname{div}^2 A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix}.$$

And then the *two* conditions $t_1 = t_2 = t_3$ are sufficient to make this quadric a sphere!

In fact, the *single* condition $(\operatorname{div} A)^2 - 3 \operatorname{div}^2 A = 0$ is sufficient to make this quadric a sphere, for

$$(\operatorname{div} A)^2 - 3 \operatorname{div}^2 A = \frac{(A_{11} - A_{22})^2 + (A_{22} - A_{33})^2 + (A_{33} - A_{11})^2}{2} + 3(A_{12}^2 + A_{23}^2 + A_{31}^2).$$

More generally, the n-dimensional (real) central hyperquadric

$$F(x) \equiv x \cdot A \cdot x + 2b \cdot x + c = 0 \quad (A = \operatorname{trn} A)$$

is a hypersphere if the coefficients of quadratic terms satisfy the *single* condition

$$(n - 1)(\operatorname{div} A)^2 - 2n \operatorname{div}^2 A = 0, \ (|A| \neq 0),$$

where

$$|t\delta - A| \equiv t^n - t^{n-1} \operatorname{div} A + t^{n-2} \operatorname{div}^2 A - \cdots + (-1)^n |A|,^1$$

i.e.

$$\operatorname{div} A = A_{11} + A_{22} + \cdots + A_{nn}, \operatorname{div}^2 A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} + \cdots + \begin{vmatrix} A_{n-1,n-1} & A_{n-1,n} \\ A_{n,n-1} & A_{nn} \end{vmatrix}.$$

Obviously it is equivalent to $(n+2)(n-1)/2$ independent conditions, as

$$(n - 1)(\operatorname{div} A)^2 - 2n \operatorname{div}^2 A = (A_{11} - A_{22})^2 + \cdots + (A_{n-1,n-1} - A_{nn})^2 + + 2n(A_{12}^2 + \cdots + A_{n-1,n}^2).$$

To use only the simplest properties of matrices on the original paradox by Prof. Coolidge[2] we have first that the three points

$$u = (u_1, u_2, u_3) \quad v = (v_1, v_2, v_3) \quad w = (w_1, w_2, w_3)$$

are collinear if, and only if, the rank of the matrix

$$M \equiv (u - v, u - w) \equiv \begin{Vmatrix} u_1 - v_1 & u_2 - v_2 & u_3 - v_3 \\ u_1 - w_1 & u_2 - w_2 & u_3 - w_3 \end{Vmatrix}$$

is one, which imposes two independent conditions on u, v, w. On the other hand, argues Prof. Coolidge (geometrically) the single condition

[1] Each coefficient of t is a Euclidean invariant of $F(x) = 0$.

[2] This MONTHLY, vol. 38, page 222.

$$\sqrt{(u-v)\cdot(u-v)}+\sqrt{(u-w)\cdot(u-w)}=\sqrt{(v-w)\cdot(v-w)}$$

is sufficient to make these three points collinear. But, squaring both sides of the last equation, we get

$$2\sqrt{(u-v)\cdot(u-v)}\sqrt{(u-w)\cdot(u-w)}=$$
$$(v-w)\cdot(v-w)-(u-v)\cdot(u-v)-(u-w)\cdot(u-w)\equiv$$
$$-2(u-v)\cdot(u-w),$$

by the law of cosines for the triangle u, v, w. Squaring again, we get

$$\begin{vmatrix}(u-v)\cdot(u-v) & (u-v)\cdot(u-w)\\ (u-v)\cdot(u-w) & (u-w)\cdot(u-w)\end{vmatrix}$$
$$\equiv|MM'|\equiv|M|\cdot|M|=0,^{1}\quad(M'=\operatorname{trn}M),$$

where $|M|$ is the vector orthogonal to $u-v$ and $u-w$ and of magnitude $|MM'|^{1/2}$ [sometimes spoken of as the vector product of $u-v$ by $u-w$ and designated by $(u-v)\times(u-w)$]. The matrix M being real, the condition $\begin{vmatrix}M\end{vmatrix}\cdot|M|=0$ (being on the left a sum of squares) requires that each term in $\begin{vmatrix}M\end{vmatrix}$ be zero, which is exactly the condition that rank $M=1$.

In n-dimensional space ($n>3$) Professor Coolidge's single condition yields, as before, $|MM'|=0$ ($M_1=u-v$, $M_2=u-w$). And, since

$$|MM'|\equiv\begin{vmatrix}u_1-v_1 & u_2-v_2\\ u_1-w_1 & u_2-w_2\end{vmatrix}^2+\begin{vmatrix}u_1-v_1 & u_3-v_3\\ u_1-w_1 & u_3-w_3\end{vmatrix}^2$$
$$+\cdots+\begin{vmatrix}u_{n-1}-v_{n-1} & u_n-v_n\\ u_{n-1}-w_{n-1} & u_n-w_n\end{vmatrix}^2,$$

the condition $|MM'|=0$ is equivalent to rank $M=1$.

A Remark on Fourier Series of Continuous Functions

By William Randels, Brown University

In studying Fourier series Riemann proved that the Fourier coefficients of an integrable function converged to zero as n increased. This theorem was later proved for Lebesgue integrable functions by Lebesgue. The next question along this line is: can anything be said about the rate of decrease of the coefficients? Hobson[2] refers to Lebesgue[3] for a proof that the theorem of Riemann-Lebesgue cannot be made more precise if one only knows that the function is continuous. Lebesgue's method is, to show that, given any function $u(n)$ such that $\lim_{n\to\infty}u(n)=0$, one may construct a step function with an infinite number of

[1] $|MM'|=|M\delta M'|=|M|\cdot adj\,\delta\cdot|M|=|M|\cdot|M|$, as $adj\,\delta=\delta$ and $x\cdot\delta=x$.

[2] Hobson, *Theory of Functions of a Real Variable*, 2nd ed., vol. 2, p. 514.

[3] Bulletin de la Société Mathématique de France, vol. 38, 1910, p. 189.

steps, and taking on only the values $+1$ and -1, whose Fourier coefficients are not of order greater than that of $u(n)$. Then he says: "By an arbitrarily small change . . . one may make $f(x)$ continuous and one will have shown that the theorem of Riemann on the decrease to zero of the Fourier coefficients of a continuous function may not be made more precise if one only knows that the function is continuous." However as the step function has an infinite number of steps, the points of discontinuity will have a limit point and the function will have a point of discontinuity of the second kind which cannot be removed by an arbitrarily small change in the function. The purpose of this note is to give a method of construction which will produce a continuous function with the desired properties.

We need the following lemma which will not be proven.

$$\lim_{n\to\infty} \int_a^b |\cos nx|\, dx = \frac{2}{\pi}(b-a). \tag{1}$$

A set of quantities l_i is defined by the relations

$$l_i = 4\pi \sum_{n=1}^{i} 5^{-n}.$$

It is obvious that $\lim_{i\to\infty} l_i = \pi$ and that $\pi - l_i = \pi/5^i$. A function $f(x)$ is assumed to be defined and continuous on the interval $(0, l_{i-1})$ and is to be defined on the interval (l_{i-1}, l_i). Then n_i is chosen so large that:

$$\left| \int_0^{l_{i-1}} f(x) \cos n_i x\, dx \right| \leqq 2 \cdot 5^{-2i} \tag{2}$$

$$\int_{l_{i-1}}^{l_i} |\cos n_i x|\, dx > \frac{7}{4\pi}(l_i - l_{i-1}) = 7 \cdot 5^{-i} \tag{3}$$

$$|u(n_i)| \leqq \tfrac{1}{3} \cdot 5^{-2i}. \tag{4}$$

The zeros of $\cos n_i x$ in the open interval (l_{i-1}, l_i) are denoted by α_{ij} $(j=1, 2, \cdots, m_i)$. Let $\delta_i = \min (5^{-i}/2(m_i+1),\ \alpha_{i1} - l_{i-1}/2,\ |\alpha_{im_i} - l_i|/2)$. Then $f(x)$ is defined as follows

$$f(x) = \begin{cases} \dfrac{5^{-i}}{\delta_i} \operatorname{sign} \cos n_i(l_{i-1} + \delta_i)(x - l_{i-1}) & (l_{i-1} \leqq x \leqq l_{i-1} + \delta_i) \\ \dfrac{5^{-i}}{\delta_i} \operatorname{sign} \cos n_i(l_i - \delta_i)(l_i - x) & (l_i - \delta_i \leqq x \leqq l_i) \\ \dfrac{5^{-i}}{\delta_i} \operatorname{sign} \cos n_i(\alpha_{ij} + \delta_i)(x - \alpha_{ij}) & (\alpha_{ij} - \delta_i \leqq x \leqq \alpha_{ij} + \delta_i) \\ 5^{-i} \operatorname{sign} \cos n_i x & \text{elsewhere on } (l_{i-1}, l_i). \end{cases}$$

As $f(x)$ is continuous on each one of the intervals (l_{i-1}, l_i) the only possible

point of discontinuity is the point π; but for every ϵ a neighborhood of the point π may be chosen such that, if x is contained in this neighborhood, $|f(x)| \leqq \epsilon$. So $f(x)$ is continuous everywhere on $(0, \pi)$. Then $f(x)$ is defined on $(-\pi, 0)$ by the relation $f(-x) = f(x)$. The n_ith Fourier coefficient of $f(x)$ is then

$$a_{n_i} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n_i x dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos n_i x dx$$

$$= \frac{2}{\pi}\left(\int_0^{l_{i-1}} f(x)\cos n_i x dx + \int_{l_{i-1}}^{l_i} f(x)\cos n_i x dx + \int_{l_i}^{\pi} f(x)\cos n_i x dx\right);$$

but we know that

$$\left|\int_0^{l_{i-1}} f(x)\cos n_i x dx\right| \leqq 2\cdot 5^{-2i}$$

and

$$\left|\int_{l_{i-1}}^{l_i} f(x)\cos n_i x dx\right| \geqq 5^i \int_{l_{i-1}}^{l_i} |\cos n_i x|\, dx - 2\cdot 5^{-i}\delta_i(m_i + 1)$$

since $f(x) = 5^{-i}$ sign $\cos n_i x$ on (l_{i-1}, l_i) except when x is within a distance δ_i of l_{i+1}, l_i or α_{ij}, where it differs from 5^{-i} sign $\cos n_i x$ by at most 5^{-i}. Also, since $|f(x)| < 5^{-i}$ on (l_i, π)

$$\left|\int_{l_i}^{\pi} f(x)\cos n_i x dx\right| < 5^{-i}(\pi - l_i) = \pi 5^{-2i}.$$

Hence

$$|a_{n_i}| > \frac{2}{\pi}(7\cdot 5^{-2i} - 5^{-2i} - 2\cdot 5^{-2i} - \pi 5^{-2i}) > \frac{5^{-2i}}{3},$$

and, by (4),

$$|a_{n_i}| > |u(n_i)|.$$

This shows that for an infinite number of values of n, a_n is greater than $u(n)$ and as $u(n)$ was perfectly arbitrary except for the condition $\lim_{n\to\infty} u(n) = 0$, this shows that the Riemann-Lebesgue theorem furnishes the best possible estimate of the order of Fourier coefficients if we only know that a function is continuous.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Foundations of Point Set Theory. (American Mathematical Society Colloquium Publications, Volume XIII.) By R. L. Moore. New York, The American Mathematical Society, 1932. vii+486 pages. $5.00 ($4.50 to members).

The book is based on a set of axioms for a space S, which make that space topologically equivalent to an ordinary two-dimensional sphere or to a plane according as S is or is not compact. As the axioms are introduced, one or two at a time, the consequences of the new axiom or axioms together with some of the old ones are developed. As a result, theorems in the early chapters apply to very wide classes of spaces, including, for example, Hilbert-space in Chapter II.

The thorough-going investigation of the properties considered is due largely to Moore and to his students. This thoroughness is evident in the present volume, which contains many results hitherto unpublished. A brief outline of the contents follows.

Chapter I. Consequences of Axioms 0 and 1. Axiom 0: *Every region is a point set.* (Point and region are undefined terms.) Axiom 1 states the existence of a sequence of collections of regions covering S, having properties related to the property of compactness. Topics treated include connectedness and separation, compactness, "Borel property," "Borel-Lebesgue property," irreducible continua, separability, properties of arcs.

Chapter II. Consequences of Axioms 1 and 2. Axiom 2: *If P is a point of a region R there exists a non-degenerate connected domain containing P and lying wholly in R.* The development leads up to necessary and sufficient conditions that a compact continuum be a "continuous curve."

Chapter III. Consequences of Axioms 1, 2, 3 and 4; and Chapter IV, Consequences of Axioms 1–5. Axiom 3: *If O is a point, $S-O$ is connected.* Axiom 4: *If J is a simple closed curve, $S-J$ is the sum of two mutually separated connected point sets such that J is the boundary of each of them.* Axiom 5: *If A is a point of a region R and B is a point distinct from A there exists, in R, a simple closed curve separating A from B.* The theorems include results in nature similar to Axioms 4 and 5; necessary and sufficient conditions that a point set be a simple closed curve, in terms of accessibility, and in other ways, and further conditions that a compact continuum be a continuous curve.

Chapter V. Upper Semi-Continuous Collections. Semi-continuity of a collection of point sets is a property defined in terms of limit points. An example of the kind of results obtained in this chapter is the following: *If K is a compact and countable point set, $S-K$ is arcwise connected.*

Chapter VI. Consequences of Axioms 1, 2, 4, 5_1, 5_2, 6 and 7; and Chapter

VII. Concerning Topological Equivalences and the Introduction of Distance. Axioms 5_1 and 5_2 are similar in nature to Axiom 5. Axiom 6: *No compact continuum separates two non-compact point sets from each other.* Axiom 7: *The set of all points is completely separable.* In Chapters VI and VII is proved what is stated in the preface, regarding the axioms named in the title of Chapter VI, ". . . every compact space that satisfies all of them is topologically equivalent to a sphere, while every non-compact one is topologically equivalent to a plane."

Of course, we have by no means referred to all the important results of the book.

The very convenient glossary on pages 485–486, giving the locations of the definitions of the technical terms of the subject, should also be mentioned.

A. B. BROWN

Plane and Spherical Trigonometry, With tables. Revised Edition. By George N. Bauer and W. E. Brooke. New York, D. C. Heath and Company, 1932. xiv+236+iv+139 pages. $2.00.

This book was first published in 1907, then revised in 1917, and again revised in 1932. When you have taught the book you realize why it has survived all these years, while other trigonometries have come and gone. It is a very teachable text book and yet it contains also a wide range of material that is useful in more advanced mathematics. This new revision of the book improves it both in content and in teachableness. However, an index would have made it handier for both student and teacher as a reference book. Also, when the authors revised the text, why did they not replace the exercises by new ones?

In this latest edition, the text has been expanded from 164 pages in length to 226 pages. No change has been made in the chapters on spherical trigonometry. A chapter on logarithms has been added, which is very necessary for the present-day student. There is a new article on reducing the expression $A \sin\theta + B \cos\theta$ to the form $C \sin(\theta+\phi)$. The graphical representation of the four fundamental operations as applied to complex numbers has been included in the chapter on De Moivre's Theorem. A four-place table of the natural trigonometric functions is now embedded in the chapter on the right triangle. More space is given to the generation of angles and to radians, short sets of examples are added at frequent intervals. Several identities are worked out in full in two ways. More space is given to graphs. More time is spent on expressing any function in terms of any other function. In a few cases alternative proofs are given of the fundamental relations between the trigonometric functions.

All this additional material improves the book without cluttering it up. It is still easy to find your way about in the text. The publishers have again produced a well-printed book in which the important topics and formulas stand out clearly.

A. D. CAMPBELL

Elementary Differential Equations. By Lyman M. Kells. New York, McGraw-Hill Book Company, 1932. ix+184 pages. $2.00.

A recital of the chapter headings and the lengths of the chapters in this book will give a good idea of the scope and nature of the text, and will show its elements of strength and of weakness. The eleven chapters are entitled as follows: Definitions and Elementary Problems (6 pages), Particular Solutions and Simple Applications (18 pages), Differential Equations of the First Order and the First Degree (21 pages), Simultaneous Equations and Problems Involving First-Order Differential Equations (3 pages), First-Order Equations of Degree Higher than the First, Singular Solutions (15 pages), Linear Differential Equations with Constant Coefficients (23 pages), Applications of Linear Equations with Constant Coefficients (16 pages), Miscellaneous Differential Equations of Order Higher than the First, Integration in Series (15 pages), Applications (9 pages), Partial Differential Equations of the First Order (18 pages), Partial Differential Equations of Order Higher than the First (17 pages). These chapters are followed by an index and a collection of answers to most of the problems. We note the small number of pages allowed each of the above subjects. In fact integration in series is summarily disposed of in two pages.

This book gives more applications of differential equations than do most similar books. We note that such topics are treated as compound-interest law problems, rates, the hodograph, harmonic motion and damping, Kirchhoff's current law and electromotive-force law and applications to networks, equation of elastic curve and deflection, along with the old standbys such as orthogonal trajectories in rectangular and in polar coordinates, envelopes, and other geometrical topics. The up-to-date character of the book is well illustrated by the section on the use of limits in place of determining the constant of integration and other constants, and also by the sections on simultaneous equations. A strong point of the book is its careful drill in setting up differential equations from given data.

The book is designed to appeal to physicists, engineers, and liberal arts students in general. For this reason the author follows a chapter on theory immediately by a chapter on applications, for he has probably often heard the question "What is the use of all this, anyhow?" Therefore the book should please the practical-minded student.

However, the brevity of the book is against it. Having fewer pages than other similar books, it covers more applications and therefore slights much theory. Even the applications are given rather too briefly. Chapter I is too short to give a student a clear idea of what the course is all about. No regular logical development is given to the subject, but rather a multitude of special types of problems are solved by ingenious but diverse methods. The discussions of the topics are not rounded off in all their details, and no wide selection of topics is offered for the teacher to choose from. No connecting link runs through the text. An intelligent student might well ask "Has every differential equation a solu-

tion? If not, why not? Just what kinds of equations can I solve in terms of well-known functions? How can I handle the other kinds of equations? What general methods of solution are there? How can I be sure that I have found all the possible solutions of any given equation?"

If the teacher is careful to supplement the text and to keep the work from becoming merely formal manipulation, he may be able to give a well-rounded elementary course in differential equations with this book as his text-book.

A. D. CAMPBELL

Georg Cantor, Gesammelte Abhandlungen. Berlin, Julius Springer, 1932. viii+486 pages. RM 48.

The scientific works of Georg Cantor are here collected, arranged and annotated by Ernst Zermelo. The mathematical papers with a few additional items—in particular appropriate extracts from the Cantor-Dedekind personal correspondence—are supplemented by a biographical sketch of 31 pages prepared by Adolf Fraenkel. A portrait serves as frontispiece. A list of seven articles on number theory and algebra is followed by nine upon trigonometric series, together filling 114 pages. The major part of the volume is of course devoted to Cantor's contributions to the theory of aggregates including the philosophy of transfinite numbers. There has been some judicious gathering under single titles of papers first appearing scattered at different times. Editorial comments are unobtrusive but highly significant, consisting of brief critical remarks, of cross-references, and of references to investigations of later mathematicians dealing with problems touched upon by Cantor.

The volume has no general index but is equipped with a serviceable index to technical terms used by Cantor in the theory of aggregates.

This book avoids giving the fragmentary impression characteristic of most editions of collected works. Neither is there the quaintness of obsolete notation. Cantor was methodical and thorough. He had few immediate pupils. He pursued so effectively the many lines of inquiry that suggested themselves in his own work that there was little left in the form of vague suggestions for pupils to pick up and develop. Such was his taste and good judgment that even in minor matters of notation, few changes have been proposed by later writers. One sees here the subject of transfinite numbers during its steady shaping at the hands of its discoverer, to a stage of well rounded symmetry. The basic concepts of Cantor are today the common property of all investigators in the theory of point sets. The more specialized notions have continued their development in the work of Hausdorff, Fraenkel, Hahn, Sierpinski, and others that require no listing here.

An elaborate appreciation of the rich material of this volume is inappropriate in this place. A trifling observation may be of interest to teachers of immature students. On looking over these pages one is impressed anew by the classical mathematical background, well-tried analytic power, and scholarly contacts of this genius whose bold originality is so startlingly conspicuous. The

hosts of undaunted circle-squarers and amateur investigators of Fermat's "last theorem," who insist that originality is to be anticipated on the part of those the soil of whose uncultivated minds is unsown to age-old prejudices, would find nothing here to support their clamor.

This book will serve a useful purpose. It is more than a gesture of homage, but the real monument to Georg Cantor is of course to be found in the subject of point sets, which he so significantly enriched.

A. A. BENNETT

Fastperiodische Funktionen. By H. Bohr. Berlin, Julius Springer, 1932. 96 pages. RM 11.40.

Almost Periodic Functions. By A. S. Besicovitch. Cambridge, University Press, 1932. xiii+180 pages. $3.75.

The monograph by Professor Bohr is one of a set of five which constitute the first volume of a new series which is being published under the direction of the editors of the "Zentralblatt für Mathematik." The new series bears the title of "Ergebnisse der Mathematik und ihrer Grenzgebiete," and is designed to supplement the efforts of the "Zentralblatt" by making available, in self-contained and concise form, expository accounts of some of the recent significant advances in the different branches of modern pure and applied mathematics.

One of the important recent developments in modern analysis has to do with the generalization of the concept of pure periodicity as a structural property of a function, and with the corresponding extensions in the theory of general trigonometric and exponential (Dirichlet) series. The class of functions to which these considerations lead, has been called "almost periodic" by Professor Bohr who initiated this line of research in 1925 in a series of memoirs published in the Acta Mathematica. The monograph under review gives a clear, self-contained exposition of some of these researches and will serve admirably as an introduction to these and subsequent investigations. It is the outgrowth of a series of lectures given by the author in 1930–31 when he was in this country as visiting professor at Stanford, the University of California and Princeton.

In order to keep the character of the work as elementary and simple as possible, the author has restricted his discussion to continuous functions of a single real variable. In the first chapter is to be found, as a necessary background for the generalizations which follow, a clear and straightforward presentation of that portion of the classical theory of periodic functions and their corresponding Fourier series, which culminates in the theorem which establishes the equivalence of the "uniqueness" theorem and the equation of Parseval. The theorem of Fejér on the summability of Fourier series and the Weierstrass approximation theorem are also given. The interrelation between all these results is shown in a direct and simple manner.

In the second chapter, almost periodicity is defined and the principal properties of almost periodic functions are established. Next, the Fourier development for such functions is defined and it is then shown that the equation of Parseval

also holds in this case, as well as its equivalence to the "uniqueness" and multiplication theorems. The chapter concludes with a proof of the important theorem which gives the fact that almost periodicity is a fundamental structural property which completely characterizes the class of functions which may be approximated uniformly by "exponential polynomials" of a certain type. Throughout these investigations the equation of Parseval plays an all important rôle. It is interesting to note that this same equation is also fundamental in the applications of the classical theory of Fourier series to geometry which were initiated by Adolf Hurwitz.

The monograph closes with two appendices, one giving a sketch of the generalizations of almost periodic functions which have been studied by Bohr, Stepanoff, Besicovitch, Weyl and others; the other is concerned with Bohr's own theory of analytic almost periodic functions and its connection with the theory of Dirichlet series.

The work of Dr. Besicovitch is more extensive and more formal in its treatment of the theory; as a whole there is relatively little overlapping with the monograph by Professor Bohr. The first chapter presents the theory of uniformly almost periodic functions in one and two real variables and follows, in the main, the methods of Bohr, de la Vallée Poussin, Weyl and Bochner. Next, there is a chapter devoted to a systematic investigation of the generalizations of the theory which have been indicated by Stepanoff, Wiener, Weyl, Bohr, Besicovitch and others. An appendix to this chapter contains some further results along these lines. The third and last chapter is given to the development of the theory of analytic almost periodic functions and their generalized Dirichlet series. Here, Bohr's work is followed in its essentials.

This authoritative work, giving the fundamental results of the theory as developed to date, will prove indispensable to any one undertaking a serious study of this interesting chapter of modern analysis.

It is to be hoped that in the not too distant future a monograph may appear dealing with the extant applications of the theory to linear differential and difference equations and with the theory of harmonic almost periodic functions developed recently (1927) by Favard in his Paris thesis.

M. A. Basoco

Modular Invariants. By D. E. Rutherford, Cambridge Mathematical Tracts, No. 27. Cambridge, University Press, 1932. vi+84 pages. $2.00.

The author of this tract has acquitted himself of a difficult task in a masterly fashion. L. E. Dickson's Madison Colloquium Lectures carried the subject of modular invariants as far as the year 1914, before the modular symbolical theory was instituted. Since then there have appeared numerous papers by Dickson, Glenn, Hazlett and others, but no complete account of the theory. The author had to select material of general import from all these papers, many of which are on special topics. He had to adopt a uniform terminology and notation. Such words as fundamental and modular, for example, vary in their

meaning from paper to paper. In Appendix I he summarizes the later papers not summarized in Dickson's History of the Theory of numbers. In Appendix II he gives a list of references. He follows the third appendix by an excellent index. The author achieves his results by giving in full some sample discussions of theory and of problems, referring to the papers for the more difficult proofs. The book is remarkably complete for so short a tract. For anyone unfamiliar with this theory of invariants we should recommend the reading of such a book as Dickson's Algebraic Invariants, followed next by Dickson's Madison Colloquium Lectures, and finally by this tract. Those who are familiar with this theory will find the tract useful as a source of information and of hints about problems for further research. For instance, the author calls attention to the fact that it has not yet been proved that all congruent covariants can be represented symbolically nor has it been shown that Miss Sanderson's theorem about invariants can be extended also to covariants.

The tract is divided into two parts. Part I deals with the general subject of modular covariants and invariants. Part II, which is taken mostly from lectures by Professor R. Weitzenbock, is given up to a fine discussion of rings and fields leading up to the proof of a theorem by E. Noether and its application to the task of proving that all the members of any system of modular covariants can be expressed in terms of a finite number of covariants.

Any mathematician, whether acquainted with the theory of modular invariants or not, will find this tract very interesting. As the author brings out so well, here we have some algebras, namely those with just a prime modulus p and also the Galois Fields, where the usual methods of attack on algebraic invariants and covariants collapse. Nothing daunted, Dickson, Glenn, Hazlett and the rest invent new and exceedingly ingenious methods. It is all a beautiful example of mathematical creative activity. Thus they cannot represent the binary cubic symbolically as α_x^3 where $\alpha_x = \alpha_1 x_1 + \alpha_2 x_2$ when the modulus $p = 3$, so they represent this cubic as $\alpha_x \beta_x \gamma_x$ where $\beta_x = \beta_1 x_1 + \beta_2 x_2$ and $\gamma_x = \gamma_1 x_1 + \gamma_2 x_2$.

Again, the operators and annihilators used in the study of algebraic invariants and covariants cannot be used for the modular types, so new operators and annihilators are invented. Dickson is especially resourceful and original in such discussions as his theory of classes and characteristic invariants and his treatment of universal covariants.

The author separates invariants and covariants into five types: algebraic, congruent where a modulus p enters in, formal where the coefficients of the transformations belong to a Galois Field, nonformal where the coefficients of the ground form belong to a Galois Field, residual where the coefficients both of the form and of the transformations belong to a Galois Field. He introduces the notation $\|$ to mean "residually congruent to" as $a^p \| a$ modulo p if a is a positive integer, and $|||$ to mean "identically congruent to" as $6 ||| 3$ modulo 3. He gives Capelli's discussion of the generators of linear transformations. He shows the relation between congruent and algebraic covariants and invariants. He discusses the pseudo-isobarism of covariants in the Galois Fields where be-

cause of Fermat's Theorem we have $T^n \| t$ (a residue in the field) and so isobarism has no meaning. He distinguishes carefully between full systems and fundamental systems and he spends much time on the question of the finiteness of a full system. We need not list any more topics to show how thoroughly he treats modular invariants in such a short space by a judicious choice of proofs and by a thrifty use of words.

A. D. CAMPBELL

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscripi should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Missouri

The officers for the academic year 1931–1932 were: Frank Cockerill, Director; R. P. Edwards, Vice Director; Robert Wier, Secretary; Mary Manley, Treasurer; Finis O. Duncan, Corresponding Secretary; Charles Clark, Librarian.

This year we had 45 active members. We awarded two prizes. The winner of the Calculus Prize was Donald Starrett Nutter and the winner of the Analytics Prize was Morris J. Gottlieb.

The meetings and programs were as follows:

September 29, 1931: At this meeting Dr. Louis Ingold gave a report of the meetings of the Mathematical Association of America and the American Mathematical Society.
October 13, 1931: "The growth of numbers" by Dr. W. D. A. Westfall.
October 27, 1931: Picnic.
November 1 , 1931: "The Phi function" by Dr. G. E. Wahlin.
November 24, 1931: "Various aids in computation" by Norman Beers and Ralph Traber.
December 1, 1931: Ciphering match (Numbers to base twelve).
December 14, 1931: Pledge examination for 10 candidates.
December 18, 1931: Formal initiation and party.
January 12, 1932: "The Mammoth cave" (Illustrated) by Dr. R. T. Dufford.
February 23, 1932: "Nomography" by John Waugh; "Vectors" by Richard Emberson.
March 8, 1932: "The algebra of logic" by Dr. Louis Ingold.
April 12, 1932: "Inequalities" by Dr. Hobart C. Carter.
April 26, 1932: Picnic.
May 10, 1932: Pledge examination for 8 candidates.
May 13, 1932: Formal initiation and banquet.

FINIS O. DUNCAN, *Corresponding Secretary*

Pi Mu Epsilon of the University of Montana

The Montana Chapter of Pi Mu Epsilon has been enjoying a very successful year. Meetings have been held regularly during the year. The officers for the year 1931–1932 were elected on November 11, 1931. They are: Franklin Long, Director; Robert Boden, Vice Director; Kathryn Coe, Secretary; Professor G. D. Shallenberger, Treasurer.

Initiation for the new members was held April 13th at the annual banquet. The following took the pledge of membership: Juanita Armour, '34; Mary Castles, '34; Roderick Chisholm, '33; Herman Dickel, '33; Lavira Hart, '33; Ruben Lewon, '33; Charlie Krebs, '34; Ellen Shields, '34; Edward Skoog, '33; Mary Rose, '32; H. A. Veeder, '32.

There are now thirty-one active members. As one of the requirements for membership, certain papers have to be prepared and presented before Pi Mu Epsilon.

The meetings and programs were as follows:

November 4, 1931: "Infinite series" by Craig Smith.
November 11, 1931: "Bell Telephone Laboratories" by Professor E. M. Little.
December 3, 1931: Applications of Calculus to Physical Chemistry" by Franklin Long.
December 10, 1931: "Temperature values" by Professor G. D. Shallenberger.
January 27, 1932: "Vector analysis" by Robert Boden.
February 17, 1932: "The quadratrix" by Robert Sullivan.
March 2, 1932: "The installment plan" by Lavira Hart.
March 16, 1932: "Rosettes" by A. L. Craig.
April 20, 1932: "The table of integrals" by William Hensen.
April 27, 1932: "Higher plane curves" by Ellen Shields and Mary Castles.
May 4, 1932: "The curve whose equation is $x^2+y^2+z^2-2xyz=0$" by John Clark.
May 18, 1932: "Hyperbolic functions" by Mary Rose.

A joint picnic with the Mathematics Club ended another successful year for the Montana Chapter of Pi Mu Epsilon.

KATHRYN COE, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematical Society of New Jersey College for Women

The officers for the year 1931–1932 were: Helen Carpenter, President; Irene Stevens, Vice President; Elizabeth Phillips, Secretary; Edith Rapp, Treasurer.

The officers are elected by ballot in the Spring usually at the last or next to last meeting of the Society for that year.

There are about twenty-four active members, that is, those who attend the meetings regularly and also take an active part in those meetings.

The primary aim of the Society is to promote a deeper interest in mathematics and to cultivate a wider knowledge in the field. To be eligible for membership one must have taken or be taking Calculus.

The meetings and the various topics discussed during the year were as follows:

October 13, 1931: "Inversion of a circle and a line" by Marian Bruen; "Inversion of angles" by Alice Maier.
October 27, 1931: "Inversion of two intersecting lines and of two intersecting planes" by Mr. Robert Walter.
November 10, 1931: "Proof that the trisection of an angle is impossible with the straight edge and compass alone" by Lucille Bourath and Anne Rosenberg.
January 12, 1932: "Construction of a seventeen sided polygon" by Mabel Joslin and Margaret McCallum.
February 16, 1932: "Constructions with a compass alone" by Armie Apamian and Edna Schnitzler.
March 1, 1932: "Brianchon's theorem" by Verna Wilson; "Kasner's pentagon theorem" by Mabel Joslin.

March 14, 1932: "The average number of sides of a polygon" by Mr. Nelson.
April 1, 1932: "A peculiar function 'A piece of Pi'" by Ethel Petrie.
April 11, 1932: "A curve that fills an area" by May Blydenburgh.
May 1, 1932: "Paradox on integration" by Molly Bruce.
May 10, 1932: "Fourier series" by Irene Stevens; "Nomography" by Marian Bruen.

At a joint meeting of the Society and the Mathematics Club of Rutgers University on December 4, 1931, Professor Alonzo Church of Princeton University lectured on "The finite geometry."

At a second joint meeting of the two organizations on April 27, 1932, Professor Thomas Fiske of Columbia University lectured on his experiences in establishing the American Mathematical Society.

Student representatives of the Society attended the meetings of the Philadelphia Section of the Mathematical Association of America at Lehigh University on Nobember 28, 1931 and the meeting of the New Jersey State Teachers Association at Atlantic City on November 28, 1931.

A bridge was given at the beginning of the year to welcome new members. A banquet was given in Professor Church's honor on December 4, 1931. A Christmas party was held just before the Christmas vacation. The annual Society dance was held on May 7, 1932.

ELIZABETH PHILLIPS, *Secretary*

Delta Chi of the University of New Hampshire

The officers for the year 1931–1932 were: William J. Volkman, President; John Grady, Vice President; Alice M. Roe, Secretary; H. Leslie Curtis, Treasurer. The officers were elected April 23, 1931 by a majority vote of the members. Alice Roe was elected secretary October 23, 1931 to serve for the year 1931–1932 because of the resignation of Margaret W. Durgin.

There are 40 active members including sophomores, juniors and seniors. Active members are those in good standing who have paid their dues. Every member of the department of mathematics is an honorary member.

The program committee which has charge of the social activities for the year consisted of Gordon Ayer, John Grady and Alice Rowe.

The objects of Delta Chi are to promote interest in the subject of mathematics and to bind the members with lasting friendship.

A student in the University of New Hampshire whose major is mathematics, or who is pursuing the Electrical Engineering, Industrial Engineering, Mechanical Engineering, Civil Engineering, or Chemical Engineering course, and who has attained an average of at least 80% or its equivalent in Mathematics 1*a* (advanced algebra), 2*b* (trigonometry), 3*c* (analytic geometry) and 7*a* (differential calculus) or their equivalent (said average to be computed by summing the four term grades and dividing by four), and a miminum of 70% each term, shall be eligible for membership.

A meetings was held January 14, 1932 with the following program:

"The theory of the planimeter" by Mr. John J. Uicker; "The rational solutions of $x^y = y^x$ (not including $x = y$, $x < 0$, or $y < 0$)" by Mr. Donald Perkins.

"Delta Chi, honorary mathematics society, will present at the end of each academic year, a silver cup to that member of the sophomore class, eligible for membership in the society, who during two year's courses in mathematics has demonstrated valuable mathematical ability, by ranking as one of the five high students in mathematics. General scholastic standing, and personality shall also figure in determining the award. A committee consisting of the Dean of the College of Technology, the Dean of the College of Liberal Arts, the head of the department of mathematics, the President of Delta Chi, and one other student member of the society shall determine the winner each year."

The trophy was won by Henry J. Joyal for the year 1930–1931. The winner for the year 1931–1932 has as yet not been determined.

Delta Chi holds two banquets each year:—the initiation banquet and the Spring banquet. The initiation banquet was held February 17, 1932 at the President's Dining Hall at the University Commons. Twenty new student members were initiated, and two new members of the mathe-

matics department, Mr. William Kichline and Mr. Milthiades Demos received honorary membership. Dr. Slobin, head of the department of mathematics, was the speaker of the evening. Mr. Moran, from the department of physics, presented a series of campus movies.

The Spring banquet was held May 18, 1932 at the Highland House in Durham. The banquet speakers were Mr. Wilbur and Mr. Solt, both members of the department of mathematics.

ALICE M. ROE, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 21. *Proposed by V. F. Ivanoff, San Francisco.*

Prove that the differences of squares of two consecutive numbers in the third diagonal of the Pascal Triangle:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, \text{etc.}$$

is always a perfect cube.

E 22. *Proposed by R. M. Winger, University of Washington.*

As a western version of problem E 7, find the digits represented by the various letters in the following problem in addition, and determine whether or not the solution is unique. (Except that obviously R and L and S and G are interchangeable.) No two different letters represent the same digit.

$$\begin{array}{cccc} S & E & N & D \\ M & O & R & E \\ G & O & L & D \\ \hline M & O & N & E & Y \end{array}$$

E 23. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

In the following sum and product, the digits from 1 through 9 are represented in some order by the letters A through I. Determine the representation and prove that it is unique.

$$AB = CD + EF, \quad EF = G \times HI.$$

E 24. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

There are just three proper fractions with denominators less than a hundred which may be reduced to lowest terms by illegitimately canceling a digit. One of these is

$$\frac{26}{65} = \frac{2\not{6}}{\not{6}5} = \frac{2}{5}.$$

Find the other two and confirm the statement that there are no others.

E 25. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

Find the only ten-place integer which is both a square and a triangular number.

SOLUTIONS

E 1. [1932, 489] *Proposed by H. E. Slaught, University of Chicago.*

Find all the unknown digits, represented by x's, in this exact division.

$$\begin{array}{l} x\ x\ x)x\ x\ x\ x\ x\ x\ x\ x(x\ 7\ x\ x\ x \\ \quad x\ x\ x\ x \\ \quad\overline{} \\ \qquad x\ x\ x \\ \qquad x\ x\ x \\ \qquad\overline{} \\ \qquad x\ x\ x\ x \\ \qquad\quad x\ x\ x \\ \qquad\quad\overline{} \\ \qquad\quad x\ x\ x\ x \\ \qquad\quad x\ x\ x\ x \\ \qquad\quad\overline{} \end{array}$$

Solution by C. A. Rupp, Pennsylvania State College.

Let D denote the divisor, and $a, 7, b, c, d$ the digits of the quotient.

1] Obviously, $c=0$.

2] Since $xx<xxx$, $xxxx-bD<xxx-7D$, and $7<b$.

3] Since $xxx<xxxx$, $bD<aD$ and $bD<dD$.

4] From [2] and [3], $7<b<a$ and $7<b<d$, and since these letters represent digits, $b=8$, $a=d=9$, and the quotient is 97809.

5] Since $8D=xxx$, $D<125$, and $9D<1125<1200$.

6] Since $xxxx-8D=xx$ (the first two digits of $9D$), it follows that $1000-8D<12$, or $123<D$. Therefore $D=124$.

7] The dividend is $124\times97809=12128316$ and the other digits of the problem follow immediately.

It is perhaps interesting to note that the problem is still soluble if the base of the number system is some integer greater than ten, the digits being 0, 1, 2, 3, $\cdots$, p, q, r, and the second digit of the quotient being p. Proceeding as

before, the quotient is $rpq0r$, $1000-12<qD<1000$, or $123+4/q<D<124+8/q$, and $D=124$ again in the new number system.

Editor's Note: Since publishing this problem, information has been received that it appeared previously in Le Sphinx.

Also solved by T. A. Bickerstaff, Annabel S. Boyce, W. E. Buker, H. E. H. Greenleaf, Arthur Haas, H. R. Leifer, A. C. Maddox, M. Markowitz, W. R. Ransom, F. C. Smith, S. Vatriquant, and B. C. Zimmerman.

E 2. [1932, 489] *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

In finding the volume cut from the sphere $x^2+y^2+z^2=9$ by the cylinder $x^2+y^2=3x$, one may use cylindrical coordinates and choose as element of volume a column of cross-section dr by $rd\theta$ and of height $z=(9-r^2)^{1/2}$. The volume appears computable either by

$$V_1 = 2\int_{-\pi/2}^{+\pi/2}\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta,$$

where symmetry with respect to the xy-plane only is used, or by

$$V_2 = 4\int_0^{\pi/2}\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta,$$

where symmetry with respect to the xz-plane is also used. But upon integration it appears that V_1 exceeds V_2 by twenty-four cubic units. Which, if either, is correct, and why?

Solution by M. Markowitz, Brooklyn, New York.

It is obvious that the volume V_1 is incorrect, since the result obtained, 18π, is the volume of a hemisphere.

To understand why

$$\int_{-\pi/2}^{0}\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta = 9\pi/2+6,$$

while

$$\int_0^{+\pi/2}\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta = 9\pi/2-6,$$

let us follow the formal steps of integrating the first integral and substituting the limits.

$$\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta = \left[-(9-r^2)^{3/2}/3\right]_0^{3\cos\theta} \tag{1}$$

$$= -(9-9\cos^2\theta)^{3/2}/3+(9)^{3/2}/3 \tag{2}$$

$$= 9-9(1-\cos^2\theta)^{3/2} \tag{3}$$

(4) $= 9 - 9\sin^3\theta.$

Now the discrepancy arises due to the transition from step (3) to step (4), for the expression $(1-\cos^2\theta)^{3/2}$ can not be negative, while its "equivalent," $\sin^3\theta$, is negative when θ is in the third or fourth quadrant. The reason that V_2 gives the correct result (as may be ascertained by using rectangular coordinates) is that the limits from 0 to $\pi/2$ do not go outside of the first quadrant. A correct result would still be obtained if the limits taken were from 0 to π.

Also solved by W. O. Cook, W. C. Janes, W. R. Ransom and D. H. Richert.

E 3. [1932, 489] *Proposed by W. R. Ransom, Tufts College.*

Has the locus, $y=x^x$, a highest point in the second quadrant?

Solution by the Proposer.

This question turns largely upon how x^x is defined. For such values as $x=2/5$ and $x=2/7$, rational fractions with even numerators and odd denominators, there certainly are points in the second quadrant, dense along a definite curve. If the definition of x^x is completed by adopting all the limiting points, we may treat x^x as a continuous curve, put $x=-z$, and locate the maximum by differentiation. This leads to $x=-1/e=-.367879$, which gives the maximum $+1.444668$ in the second quadrant. But it is questionable whether this point may be regarded as present on the curve according to the usual convention as to powers. For the device by which it is made one of the points of the curve, demands that we acknowledge $(-3)^{-3}$ to have the value $+1/27$ as well as the value $-1/27$.

Also solved by W. C. Janes.

E 4. [1932, 489] *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

What is the simplest way to cut a wooden block 1 ft. $\times$ 1 ft. $\times$ 2 ft. into pieces which may be reassembled into a solid cube?

Solution by W. R. Ransom, Tufts College.

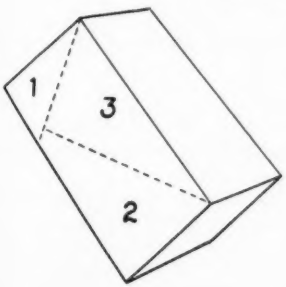


FIG. I.

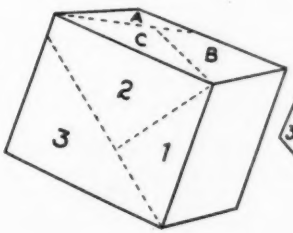


FIG. II.

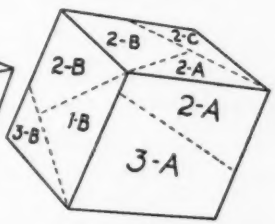


FIG. III.

A wooden block 1 ft. $\times$ 1 ft. $\times$ 2 ft. may be cut into eight pieces by four plane cuts and reassembled into a solid cube. The original block and the first two cuts are shown in figure I. Piece 1 is cut off by a plane cut through one of the shorter edges, the length of the cut being $2^{1/3}$. The other cut in this first figure is made by a plane perpendicular to the first cut, and through the edge farthest from the first cutting plane. The resulting pieces, 1, 2 and 3, are then reassembled as shown in figure II, the dimensions of which block are 1 ft. $\times 2^{1/3}$ ft. $\times 2^{2/3}$ ft. Two plane cuts are now made perpendicular to the two opposite faces of medium area. The first of these, cutting off piece A, goes through one edge of the reassembled block and is of length $2^{1/3}$. This plane dissects pieces 2 and 3, but does not touch piece 1. The final cutting plane is perpendicular to this last previous plane, and again through an edge of the reassembled block. It dissects pieces 1, 2 and 3. Thus we have piece 1 separated into two pieces, while pieces 2 and 3 are separated into three pieces each, giving us eight pieces in all. Finally, if the pieces from figure II are rearranged as shown in figure III, the result is a cube, each dimension being $2^{1/3}$.

Note: Since this solution was sent to the printer, the author has learned of a solution of the problem using *only seven* pieces, by A. H. Wheeler of Worcester, Mass.

Also solved by the proposer.

E 5. [1932, 489] *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

How may the total surface of a sphere be divided into the largest possible number of congruent pieces, if each side of each piece is an arc of a great circle less than a quadrant?

Solution by W. R. Ransom, Tufts College.

Inscribe a regular dodecahedron or a regular icosahedron in the sphere and drop perpendiculars from the center to each face. Form isosceles triangles, sixty in number, with the feet of these perpendiculars as vertices and the sides of the respective faces as bases, and centrally project these sixty triangles onto the surface of the sphere as isosceles spherical triangles.

Also solved by the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3594. *Proposed by H. T. R. Aude, Colgate University.*

Find sets of integers for rational right triangles which, as the numbers increase, approach a 30°–60° right triangle.

3595. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Through the edges a, b, c, of a trihedral angle planes are drawn perpendicular to the faces bc, ca, ab and cutting these faces along the lines e, f, g, respectively. Show that the faces of the given trihedral angle bisect the dihedral angles of the trihedral angle formed by the lines e, f, g.

3596. *Proposed by L. S. Johnston, University of Detroit.*

Given

$$ {}_sH_r = \frac{s(s+1)(s+2)\cdots(s+r-1)}{r!}, \quad {}_sC_r = \frac{s(s-1)(s-2)\cdots(s-r+1)}{r!}, $$

where s and r are any positive integers; and

$$ {}_sH_0 = {}_sC_0 = 1 $$

where s is any positive integer: prove that

$$ \phi(s, p) = \sum_{k=0}^{p} (-1)^k {}_sH_{p-k}\, {}_sC_k = 0 $$

for all positive integral values of s and p.

3597. *Proposed by J. E. Trevor, Cornell University.*

Continuous functions $f(x, y)$ and $F(x, y)$ of the independent variables x, y have continuous and non-vanishing first and second derivatives p, q, r, s, t and P, Q, R, S, T respectively. The surface $z=f(x, y)$ is tangent to the *developable surface* $Z=F(x, y)$ along a line whose projection on the x, y plane is a curve $\phi(x, y)=0$. On the line of contact x and y are functions of q. Writing dx/dq and dy/dq for the first derivatives of these functions, and putting $\Delta=rt-s^2$, show that, at points on the line of contact,

$$ r - R = + T\Delta\left(\frac{dy}{dq}\right)^2 $$

$$ s - S = - T\Delta\frac{dx}{dq}\frac{dy}{dq} $$

$$ t - T = + T\Delta\left(\frac{dx}{dq}\right)^2. $$

This problem is the mathematical aspect of a problem in thermodynamics.

3598. *Proposed by V. Thébault, Le Mans, France.*

On the lines PA, PB, PC, PD, joining a point P in space to the vertices of the *orthocentric* tetrahedron $ABCD$ are marked the inverse points A_1, B_1, C_1, D_1, of the vertices in the inversion (P, k). The planes perpendicular at these points to the lines PA_1, PB_1, PC_1, PD_1, form a tetrahedron $\alpha\beta\gamma\delta$. Show that the planes perpendicular at P to the lines $P\alpha$, $P\beta$, $P\gamma$, $P\delta$, cut the planes $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$, $\alpha\beta\gamma$ along four coplanar lines, and that the plane of these lines is perpendicular to the line joining P to the orthocenter of the tetrahedron $ABCD$.

3599. *Proposed by Roy MacKay, Albuquerque, New Mexico.*

If $f(0) = 1$ and

$$f(x) = \left(\frac{x}{e^x - 1}\right)^k \quad \text{for} \quad x \neq 0,$$

where k is a positive integer $\geqq 2$; prove that

$$\left[\frac{d^r}{dx^r} f(x)\right]_{x=0} = \frac{(-1)^r \sigma_r}{\binom{k-1}{r}},$$

where r is any positive integer $\leqq k-1$; and σ_r is the elementary symmetric function of the numbers $1, 2, 3, \cdots, (k-1)$, that is σ_r is the sum of the products of these numbers taken r at a time.

3600. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

If α and β are the angles which the axis s of a cone of revolution makes with the H and V planes of projection respectively; ϵ the apex angle of the cone; a_1 and b_1 the major and minor semi-axes of the elliptical H-section of the cone; a_2 and b_2 the major and minor semi-axes of the elliptical V-section; θ the angle which the H-projection of s makes with the line HV of intersection of H and V: show that

$$\cos\frac{\epsilon}{2} = \frac{a_1 a_2}{\left\{a_1^2(a_2^2 - b_2^2) + a_2^2(a_1^2 - b_1^2)\sin^2\theta\right\}^{1/2}}$$

$$\cos\beta = \frac{a_1(a_2^2 - b_2^2)^{1/2}}{\left\{a_1^2(a_2^2 - b_2^2) + a_2^2(a_1^2 - b_1^2)\sin^2\theta\right\}^{1/2}}$$

$$\cos\alpha = \frac{a_2(a_1^2 - b_1^2)^{1/2}}{\left\{a_1^2(a_2^2 - b_2^2) + a_2^2(a_1^2 - b_1^2)\sin^2\theta\right\}^{1/2}}$$

SOLUTIONS

3513. [1931, 461]. *Proposed by L. S. Johnston, University of Detroit.*

Given

$$f(s, p) = \sum_{k=1}^{p}(-1)^{p+k}(2k-1)\,{}_sH_{p-k}$$

where

$${}_sH_r \equiv \frac{s(s+1)(s+2)\cdots(s+r-1)}{r!}, \quad {}_sH_0 \equiv 1.$$

Prove: (a) If p is odd, then $f(s, p)$ is positive for all values of s;

(b) If p is even, then $f(s, p)$ is positive, equal to zero, or negative according as s is less than, equal to, or greater than 3.

Solution by Frank Ayres, Jr., Dickinson College

The function $f(s, p)$ may be given a different form by writing $2k-1=k-1+k$ and then using the relation ${}_sH_n-{}_sH_{n-1}={}_{s-1}H_n$. There results

$$f(s, p) = (-1)^{p-1}\sum_{k=1}^{p}(-1)^{k-1}k\,{}_{s-1}H_{p-k}. \tag{1}$$

Now using a similar reduction after separating the even and odd terms, we find that

$$f(s, p) = (-1)^{p-1}\sum_{i=1}^{q} i\,{}_{s-3}H_{p-2i+1}, \tag{2}$$

where q is the greatest integer in $(p+1)/2$.

If p is odd, $2m+1$, $f(s, p)$ is positive for all real values of s except possibly in the intervals

$$4-2j < s < 5-2j, \quad j = 1, 2, \cdots, m. \tag{3}$$

If p is even, $2m$, $f(s, p)$ is negative for all real values of $s>3$; it is zero for $s=3$; it is positive for all other real values of s except possibly in the intervals

$$3-2j < s < 4-2j, \quad j = 1, 2, \cdots, m-1. \tag{4}$$

Further investigation is needed to determine the sign of $f(s, p)$ in the intervals (3) and (4).

A Note by the Editors. Since the printing of this problem it has been learned from the proposer that s is restricted to integral values, positive, negative, or zero. With this restriction the above solution is complete, as no integral values of s lie in the intervals (3) and (4).

Also solved by the proposer.

3527. [1932, 46]. *Proposed by R. E. Gaines, The University of Richmond.*

If a variable chord QR of a conic subtends a right angle at a fixed point P on the conic, the chord QR passes through a fixed point S. If P describes the

conic, the locus of S is a conic whose equation differs from the equation of the given conic only in the constant term.

Solution by Roscoe Woods, University of Iowa.

The fixed point S in the above problem is known as Frégier's point and is situated upon the normal to the conic at P. See Sommerville, *Analytical Conics* (1924), p. 104. This problem for the ellipse is stated and proved in Casey, *A Treatise on Analytical Geometry*, etc. (1893), p. 227.

The analytical solution of this problem can be put in very compact form as follows: By the proper choice of the axes the equation of the general conic takes the form

$$ax^2 + by^2 - 2gx = 0, \quad bg \neq 0. \tag{1}$$

Let $P(x_1, y_1)$ be any point on the conic in equation (1). Since every chord QR which subtends a right angle at P cuts the normal at P in a fixed point, S, we choose the two perpendicular chords $x=x_1$ and $y=y_1$, and find (if $a\neq 0$) the coordinates of Q and R to be $[(2g-ax_1)/a, y_1]$ and $(x_1, -y_1)$, respectively. Solving the equation of the chord QR with the normal at P simultaneously, we find the coordinates of S to be

$$\left[\frac{(b-a)x_1+2g}{a+b}, \frac{(a-b)y_1}{a+b}\right].$$

It is easily verified that this result holds also for $a=0$. The elimination of x_1 and y_1 from the coordinates of S by means of the equation (1) gives the locus of S to be

$$ax^2 + by^2 - 2gx + 4g^2b/(a+b)^2 = 0$$

which demonstrates the theorem.

A Note by the Editors. A proof of the fact that QR passes through a fixed point S is given by Goormaghtigh [1929, 113] in his solution of a generalization of problem 3302 [1928, 41]. The point S must lie on the normal to the conic at P, for, when PQ is tangent to the conic, QR coincides with the normal at P. An exception must be made in the case of an equilateral hyperbola, $a+b=0$. In this case all the chords QR are parallel.

Also solved by E. F. Allen, Theodore Bennett, J. H. Butchart, R. Goormaghtigh, V. F. Murray, and Paul Wernicke.

3529. [1932, 115]. *Proposed by V. F. Ivanoff, San Francisco, California.*

A ship is sailing with a speed and direction, $\boldsymbol{v}_1$; the wind blows apparently (judging by the vane on the mast) in the direction of a vector, $\boldsymbol{a}$; on changing the direction and the speed of the ship from $\boldsymbol{v}_1$ to $\boldsymbol{v}_2$, the apparent wind is in the direction of a vector, $\boldsymbol{b}$.

Find the vector velocity of the wind.

Solution by T. C. Esty, Amherst College

The actual velocity, $\boldsymbol{v}$, of the wind is the sum of the ship's velocity and the apparent velocity of the wind. Without loss of generality we may assume $\boldsymbol{a}$ and $\boldsymbol{b}$ to be unit vectors, and may write

$$\boldsymbol{v} = \boldsymbol{v}_1 + s\boldsymbol{a} = \boldsymbol{v}_2 + t\boldsymbol{b}, \tag{1}$$

where s and t are undetermined scalars. Multiply (1) scalarly first by $\boldsymbol{a}$ and then by $\boldsymbol{b}$, and obtain

$$s - t\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}\cdot(\boldsymbol{v}_2 - \boldsymbol{v}_1), \tag{2}$$

$$s\boldsymbol{a}\cdot\boldsymbol{b} - t = \boldsymbol{b}\cdot(\boldsymbol{v}_2 - \boldsymbol{v}_1). \tag{3}$$

Multiply (3) by $\boldsymbol{a}\cdot\boldsymbol{b}$, subtract the result from (2), and find

$$s = \frac{[\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{b}]\cdot(\boldsymbol{v}_2 - \boldsymbol{v}_1)}{1 - (\boldsymbol{a}\cdot\boldsymbol{b})^2}.$$

Substituting in (1) we get

$$\boldsymbol{v} = \boldsymbol{v}_1 + \frac{[\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{b}]\cdot(\boldsymbol{v}_2 - \boldsymbol{v}_1)\boldsymbol{a}}{1 - (\boldsymbol{a}\cdot\boldsymbol{b})^2}.$$

Also solved by R. P. Agnew, D. F. Gunder, A. Pelletier, H. D. Ruderman, H. L. Schug, and Paul Wernicke.

3530. [1932, 116] *Proposed by A. A. Shaw, University of Arizona.*

Prove that the product of any three consecutive integers is a multiple of 504, if the middle integer is a perfect cube.

Solution by Lester R. Ford, The Rice Institute

The product has the form $(n^3-1)n^3(n^3+1)$, or n^9-n^3, where n is an integer. We show that this is divisible by the relatively prime integers 7, 8, and 9, whose product is 504.

We have

$$n \equiv 0, \pm 1, \pm 2, \pm 3 \pmod 7.$$

Cubing and reducing modulo 7,

$$n^3 \equiv 0, \pm 1 \pmod 7.$$

Cubing again,

$$n^9 \equiv 0, \pm 1 \pmod 7.$$

Hence

$$n^9 - n^3 \equiv 0 \pmod 7.$$

In a similar manner

$$n \equiv 0, \pm 1, \pm 2, \pm 3, 4 \pmod 8,$$
$$n^3 \equiv 0, \pm 1, \pm 3 \pmod 8,$$
$$n^9 \equiv 0, \pm 1, \pm 3 \pmod 8,$$
$$n^9 - n^3 \equiv 0, \pmod 8;$$

and

$$n \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4 \pmod 9,$$
$$n^3 \equiv 0, \pm 1 \pmod 9,$$
$$n^9 \equiv 0, \pm 1 \pmod 9,$$
$$n^9 - n^3 \equiv 0 \pmod 9;$$

and the proposition is established.

Also solved by R. P. Agnew, B. R. Allen, E. Alliot, H. T. R. Aude, Brother Aurelius, Frank Ayres, W. E. Buker, Thomas Bulter, S. S. Cairns, Mannis Charosh, J. A. Clarkson, O. C. Collins, G. F. Cramer, J. W. Foust, H. M. Gehman, R. Goormaghtigh, D. F. Gunder, J. D. Hill, V. F. Ivanoff, Sister Mary Bonaventure, A. S. Merrill, Dave Montgomery, R. E. Moritz, A. Pelletier, W. M. Rust, M. A. Scheier, L. S. Shively, H. W. Smith, R. C. Staley, Mildred E. Taylor, F, Underwood, Paul Wernicke, T. R. C. Wilson.

3531. [1932, 116]. *Proposed by Eugene M. Berry, Lynchburg College.*

We have given a horizontal bar of length b, with a (vertical) leg at each end of the bar. The legs are short and of equal length. The front leg comes to a point at the bottom while the foot of the rear leg is a knife edge which is parallel to the bar. If this is placed with the legs on paper find the curve traced by the knife edge when the front leg traces a given circle.

Solution by D. F. Gunder, Colorado Agricultural College

From the given conditions we deduce the following relations, in which x, y and x_1, y_1 are respectively the coordinates of points on the required curve and the given circle.

$$(y - y_1)^2 + (x - x_1)^2 = b^2$$
$$(y - y_1) - \frac{dy}{dx}(x - x_1) = 0$$
$$x_1^2 + y_1^2 = a^2.$$

By elimination of x_1 and y_1 from these, the differential equation of the curve is found to be:

$$(x^2 + y^2 + b^2 - a^2)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 4b^2\left[y\frac{dy}{dx} + x\right]^2.$$

In polar coordinates this gives the first order equation,

$$d\rho\sqrt{4b^2\rho^2 - (\rho^2 + b^2 - a^2)^2} = \rho(\rho^2 + b^2 - a^2)d\theta,$$

which by quadrature yields the solution:

$$\theta + c = \frac{-b}{\sqrt{b^2 - a^2}} \tan^{-1}\left\{\frac{\sqrt{b^2 - a^2}}{b} \cdot f(\rho)\right\} + \tan^{-1} f(\rho),$$

where

$$f(\rho) = \frac{\sqrt{4b^2\rho^2 - (\rho^2 + b^2 - a^2)^2}}{\rho^2 - b^2 + a^2}.$$

Applying the initial condition that $\rho = b+a$, $\theta = 0$, gives $c = 0$, $\rho > 0$. The latter condition, $\rho > 0$, eliminates the initial condition $\theta = 0$, $\rho = -b-a$ or $-b+a$, which is otherwise contained in the solution when $c = 0$. The other initial condition, $\theta = 0$, $\rho = b-a$, which also gives $c = 0$ must also be excluded.

With these restrictions due to the initial conditions, which are general, the curve is briefly discussed as follows:

If $b > a$, the equation is found to represent a saw-toothed curve whose farthest points from the pole lie on a circle of radius $b+a$, the angle subtended at the pole by the arc determined by two consecutive extreme points being $[b(b^2-a^2)^{-1/2}-1]2\pi$. The nearest points of the curve lie on a circle of radius $b-a$ at the same angular intervals as above. All extreme points are at cusps of the curve. The curve repeats itself after p revolutions if $b(b^2-a^2)^{-1/2}-1 = p/q$, where p and q are relatively prime integers.

If $b < a$, the first term of the right hand member of the solution reduces to

$$\frac{b}{\sqrt{a^2 - b^2}} \tanh^{-1} \frac{\sqrt{a^2 - b^2}}{b} f(\rho),$$

and the solution represents a spiral, winding about and approaching a circle of radius $a-b$. If $b = a$, a spiral about the pole is obtained.

A Note by the Editors. The differential equation in polar coordinates may be obtained more directly from the formula $\tan \phi = \rho d\theta/d\rho$, where ϕ is the angle opposite a in the triangle whose sides are a, b, ρ. By the use of the law of cosines ϕ may be eliminated and we obtain the desired equation. It is easily seen that the center of curvature is the intersection of the normal to the curve with the corresponding radius of the circle. The solution and initial conditions given above are not quite general. For the initial conditions $\theta = 0$, $\rho = \sqrt{a^2-b^2}$, it is readily seen that the solution is the circle $\rho = \sqrt{a^2-b^2}$.

3532 [1932, 116]. *Proposed by R. E. Gaines, University of Richmond.*

If the triangle PQR is inscribed in an ellipse so as to cut off three segments of equal area, and tangents are drawn at P, Q, R, the triangle thus formed will be inscribed in a second ellipse which will be divided in the same way.

Solution by Roscoe Woods, University of Iowa

Consider a circle of radius a with center at the origin. Let pqr be any equilateral triangle inscribed in the circle. It cuts off three segments of equal area. The tangents at the vertices p, q, r form a second equilateral triangle inscribed in a second circle of radius $2a$ and consequently cut off three segments of equal area from the second circle.

Let us subject this configuration to an orthogonal projection given by the equations $x=x'$, $y=ay'/b$. The triangles project into triangles and the circles project into ellipses. The area of the triangle pqr is in a constant ratio, a/b, to the area of its projection PQR. The same is true of any two corresponding areas. This is easily seen from the transformations above. Hence the triangle PQR cuts off three equal area segments from the ellipse. The same can be said about the triangle formed by the tangents at the vertices P, Q, R.

The triangle PQR is a triangle of maximum area inscribed in the ellipse. Since there is a single infinity of equilateral triangles inscribed in the first circle, there is a single infinity of triangles inscribed in the ellipse that cut off three segments of equal area in the ellipse. See, C. Smith, *Conic Sections*, p. 173.

Also solved by R. Goormaghtigh, V. F. Murray, O. J. Ramler, and F. Underwood.

3533 [1932, 116]. *Proposed by R. E. Gaines, University of Richmond.*

The generating circle of a cycloid in one position is tangent to one arch of the cycloid and intersects the next following arch in P_1 and P_2; another such circle intersects the same arch in P_2 and P_3; another, in P_3 and P_4; and so on indefinitely. If $\alpha_1, \alpha_2, \alpha_3, \cdots$ are the angles which the chords P_1P_2, P_2P_3, $P_3P_4 \cdots$ make with the base of the cycloid, then

$$6\sum_{n=1}^{\infty}\alpha_n = 4\pi - 3^{3/2}.$$

Solution by Wm. B. Campbell, Rangoon, Burma

When the unit generating circle has its center at $(\theta_1, 1)$, the point P_1 of this circle generating the cycloid has the coordinates $(\theta_1-\sin\theta_1, 1-\cos\theta_1)$; and the circle cuts the cycloid again in P_2, with the parameter θ_2 which satisfies the equation

$$(\theta_2 - \sin\theta_2 - \theta_1)^2 + (1 - \cos\theta_2 - 1)^2 = 1, \text{ or}$$
$$\sin\theta_2 = \tfrac{1}{2}(\theta_2 - \theta_1). \tag{1}$$

If P_1 is on the first half arch, that is, if $0<\theta_1<\pi$, then $\theta_1<\theta_2<\pi$. If α_1 is the inclination of the line P_1P_2,

$$\tan\alpha_1 = \frac{\cos\theta_1 - \cos\theta_2}{\theta_2 - \theta_1 + \sin\theta_1 - \sin\theta_2} = \tan\tfrac{1}{2}(\theta_2 - \theta_1),$$

where the reduction follows from (1). Since α_1 and $\frac{1}{2}(\theta_2-\theta_1)$ are in the first quadrant, we have

$$\alpha_1 = \tfrac{1}{2}(\theta_2 - \theta_1). \tag{2}$$

Similarly, we have for P_2P_3, $\alpha_2=\frac{1}{2}(\theta_3-\theta_2)$, etc. The θ's form an increasing sequence which approaches a limit $\leqq\pi$, and (1) shows that this limit is precisely π. Hence

$$\sum_{n=1}^{\infty}\alpha_n = \tfrac{1}{2}(\pi - \theta_1). \tag{3}$$

If the first circle with center $(\theta_1, 1)$ is tangent at P_0 to the preceding arch of the cycloid, the common normal at P_0 has the slope $-(1-\cos\theta_0)/\sin\theta_0 = -\tan\frac{1}{2}\theta_0$. Here $-\pi<\theta_0<0$, and we find for $(\theta_1, 1)$ the values

$$\theta_0 - \sin\theta_0 + \cos\tfrac{1}{2}\theta_0,\ 1 - \cos\theta_0 - \sin\tfrac{1}{2}\theta_0.$$

The resulting two equations give $\theta_0=-\pi/3$, $\theta_1=3^{1/2}-\pi/3$; whence

$$6\sum_{n=1}^{\infty}\alpha_n = 3(\pi - \theta_1) = 4\pi - 3^{3/2}.$$

A Note by Otto Dunkel. The required relations may be easily obtained from a figure. Let the rolling circle with center C_1 be tangent to the base at M_1 and tangent externally to the preceding arch at P_0. Let the circle with center C_0 which passes through P_0 and determines that point be tangent to the base at M_0. Since M_0 is the instantaneous center of rotation for P_0 on the circle, M_0P_0 is normal at P_0 to the preceding arch, and M_0P_0 passes through C_1. In the rectangle $M_0M_1C_1C_0$, $C_0P_0=C_1P_0=\frac{1}{2}M_0C_1$; hence the triangles $M_0P_0C_0$ and $M_1P_0C_1$ are equilateral. We have then $M_0O=\pi/3=-\theta_0$, $M_0M_1=3^{1/2}$. The circle C_1 cuts the cycloid again in the nearer point P_1 and in the more distant point P_2, where $\theta_1=\text{arc } M_1P_1=OM_1=3^{1/2}-\pi/3$. Let C_2 be the point on C_0C_1 extended so that $P_2C_2=P_2C_1$; then C_2 is the center of the generating circle for P_2. Let this circle be tangent to the base at M_2. We see from the figure that $M_1M_2=\theta_2-\theta_1=2\sin\theta_2$, and that $\angle M_1C_1P_2=\angle M_2C_2P_2=\theta_2<\pi$. If P_1' on circle C_1 is symmetric to P_1 with respect to M_1C_1, then $\angle P_1'C_1P_2=\theta_2-\theta_1$, and hence $\angle P_1'P_1P_2=\alpha_1=\frac{1}{2}(\theta_2-\theta_1)$. The rest easily follows.

A solution of 3488 by Mannis Charosh was overlooked in the list of solvers [1932, 242]. A solution of 3483 by A. E. Fuller was received too late for credit at the proper time.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ooio.

Mr. R. C. Bullock, of the University of North Carolina, has been appointed to an acting professorship at Lambuth College, Jackson, Tenn.

Dr. Fay Farnum has been promoted to an assistant professorship, at Washington Square College, New York University.

Dr. Eberhard Hopf has been appointed assistant professor of mathematics at the Massachusetts Institute of Technology.

Professor Alfred Hume, of the University of Tennessee, has been reinstated as chancellor of the University of Mississippi.

Assistant Professor F. W. John has been promoted to an associate professorship at Washington Square College, New York University.

Assistant Professor A. L. Meder, of the department of mathematics at the New Jersey College for Women, has been appointed acting dean of that college.

Professor O. M. Norlie, of Hartwick College, has been appointed director of the new graduate school of Hartwick Seminary, Brooklyn.

Dr. Jerry H. Service, of Henderson State College, has been appointed assistant professor of mathematics at Michigan College of Mining and Technology.

Dr. W. R. Thompson has been appointed research assistant in pathology at Yale University.

Assistant Professor Marion M. Torrey, of Goucher College, has been promoted to an associate professorship.

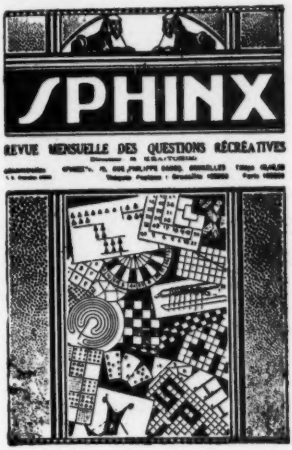
Rev. Joseph Wilczewski, of St. Louis University, has been appointed to a professorship at Mount St. Michael.

Miss Marguerite L. Zeigel has been appointed to a position in the department of mathematics and physics at Lander College, Greenwood, S.C.

The following appointments to instructorships have been made:

Long Beach Junior College, Mr. A. L. Buckman.
Michigan College of Mining and Technology, Mr. Nels Johnson.
University of Minnesota, Dr. C. H. Fischer.
New York University, University Heights Branch, Mr. F. C. Hall.
Tulane University, Mr. J. F. Thomson.

Professor R. L. Green, professor emeritus of mathematics at Stanford University, died on Nov. 19, 1932, in his seventy-first year. He was a charter member of the Mathematical Association.



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THE CARUS MONOGRAPH COMMITTEE is pleased to announce that the first edition of Number Four is well advanced in sales and that each of the others has gone into a second edition; also that a German Edition of Number One is being brought out by the firm of Teubner in Leipzig and Berlin. The titles of the monographs are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS; (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ; "Projective Geometry," by Professor JOHN W. YOUNG.

The price of these Monographs is $1.25 per copy to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 337 East Chicago Avenue, Chicago, Illinois, distributors to the general public of Association publications.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

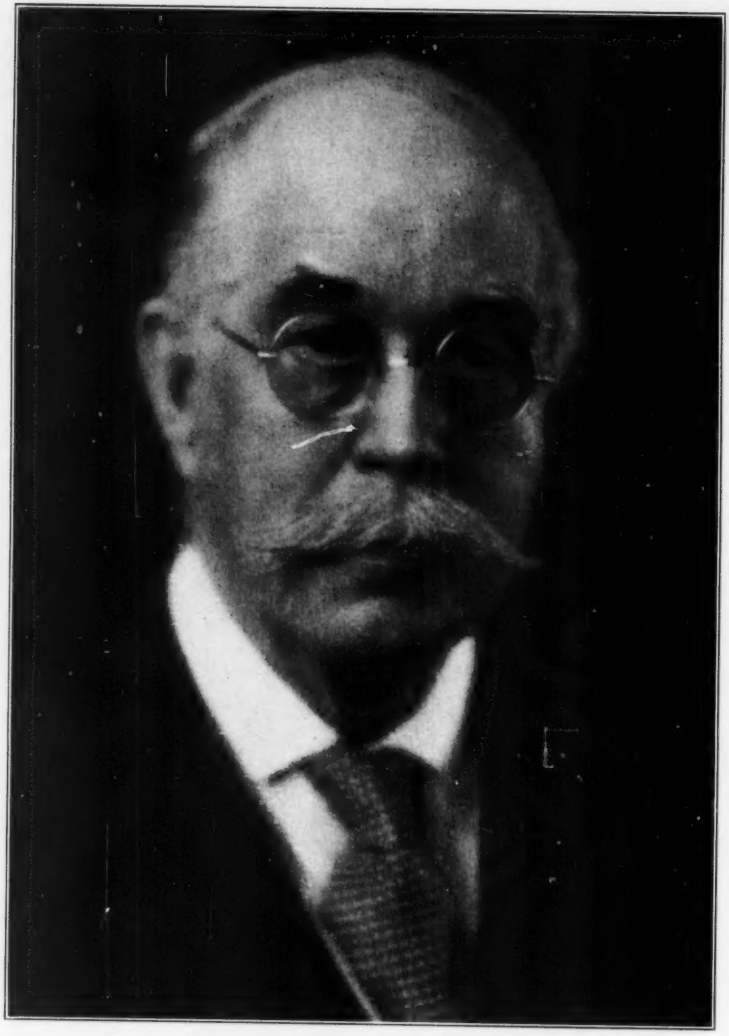
MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Summer Meeting of the Association, Chicago, Ill., June 20-22, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, April.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.



Arnold Fuffum Chace, 1845-1932

THE SEVENTEENTH ANNUAL MEETING OF THE ASSOCIATION

The seventeenth annual meeting of the Mathematical Association of America was held at Atlantic City, New Jersey, on Tuesday and Wednesday, December 27 and 28, 1932, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. Two hundred eighty-nine were in attendance at the meetings, including the following one hundred seventy-seven members of the Association:

C. R. Adams, Brown University
V. W. Adkisson, University of Arkansas
R. P. Agnew, Cornell University
Beatrice Aitchison, Johns Hopkins University
R. C. Archibald, Brown University
H. E. Arnold, Wesleyan University
C. S. Atchison, Washington and Jefferson College
Frank Ayres, Jr., Dickinson College

R. W. Babcock, Kansas State Agricultural College
Clara L. Bacon, Goucher College
N. H. Ball, Princeton University
Ethelwynn R. Beckwith, Milwaukee-Downer College
A. A. Bennett, Brown University
G. D. Birkhoff, Harvard University
Archie Blake, U. S. Coast and Geodetic Survey
Henry Blumberg, Ohio State University
Joseph Bowden, Adelphi College
Julia W. Bower, University of Chicago
N. R. Bryan, University of Maine
R. W. Burgess, Western Electric Company
J. H. Bushey, Hunter College
W. H. Bussey, University of Minnesota
W. E. Byrne, Virginia Military Institute

S. S. Cairns, Lehigh University
W. D. Cairns, Oberlin College
R. H. Cameron, Cornell University
B. H. Camp, Wesleyan University
P. A. Caris, University of Pennsylvania
Mildred E. Carlen, Brown University
W. B. Carver, Cornell University
G. G. Chambers, University of Pennsylvania
W. F. Cheney, Jr., Connecticut Agricultural College
A. B. Coble, University of Illinois

Abraham Cohen, Johns Hopkins University
J. B. Coleman, University of South Carolina
T. F. Cope, Marietta College
Lennie P. Copeland, Wellesley College
C. H. Currier, Brown University
E. H. Cutler, Lehigh University

Tobias Dantzig, University of Maryland
Marguerite D. Darkow, Hunter College
H. T. Davis, Indiana University
J. E. Davis, Drexel Institute
F. F. Decker, Syracuse University
C. E. Dimick, U. S. Coast Guard Academy
H. A. DoBell, New York State College for Teachers
J. L. Dorroh, Princeton University
B. F. Dostal, University of Florida
Arnold Dresden, Swarthmore College
Melvin Dresher, Lehigh University
L. A. Dye, Cornell University

L. P. Eisenhart, Princeton University
G. C. Evans, The Rice Institute
G. W. Evans, Lynn, Mass.
H. S. Everett, University of Chicago

F. J. Feinler, St. Peter's Church, Loudonville, Ohio
Irving Fisher, Yale University
M. M. Flood, Princeton University
Tomlinson Fort, Lehigh University
T. C. Fry, Bell Telephone Laboratories

A. S. Gale, University of Rochester
F. J. Gerst, Loyola University, Chicago, Ill.
D. C. Gillespie, Cornell University
J. S. Gold, Bucknell University
Michael Goldberg, Bureau of Ordnance, Navy Dept., Washington, D.C.
H. V. Gummere, Haverford College
Harry Gwinner, University of Maryland